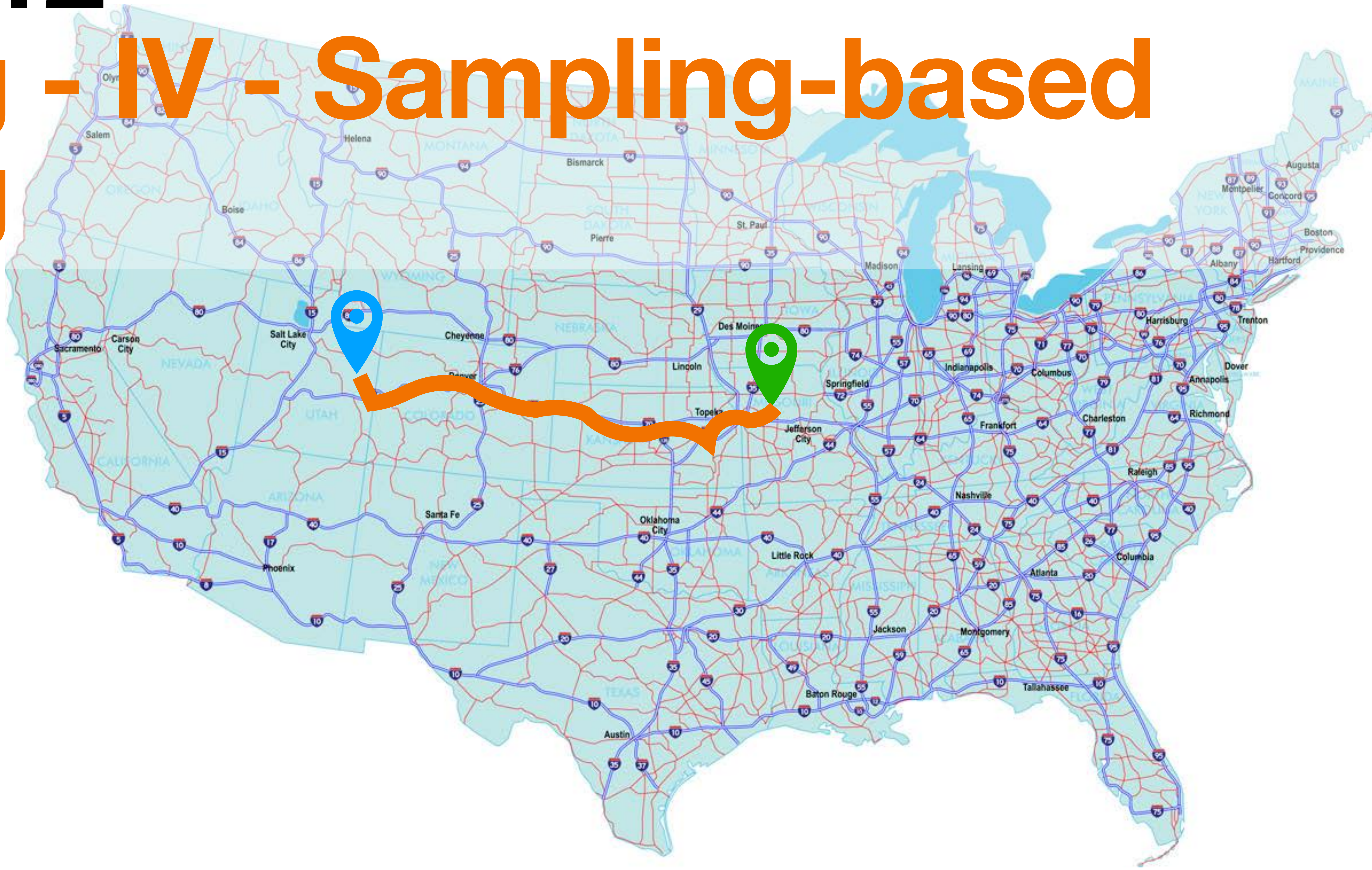
Lecture 12

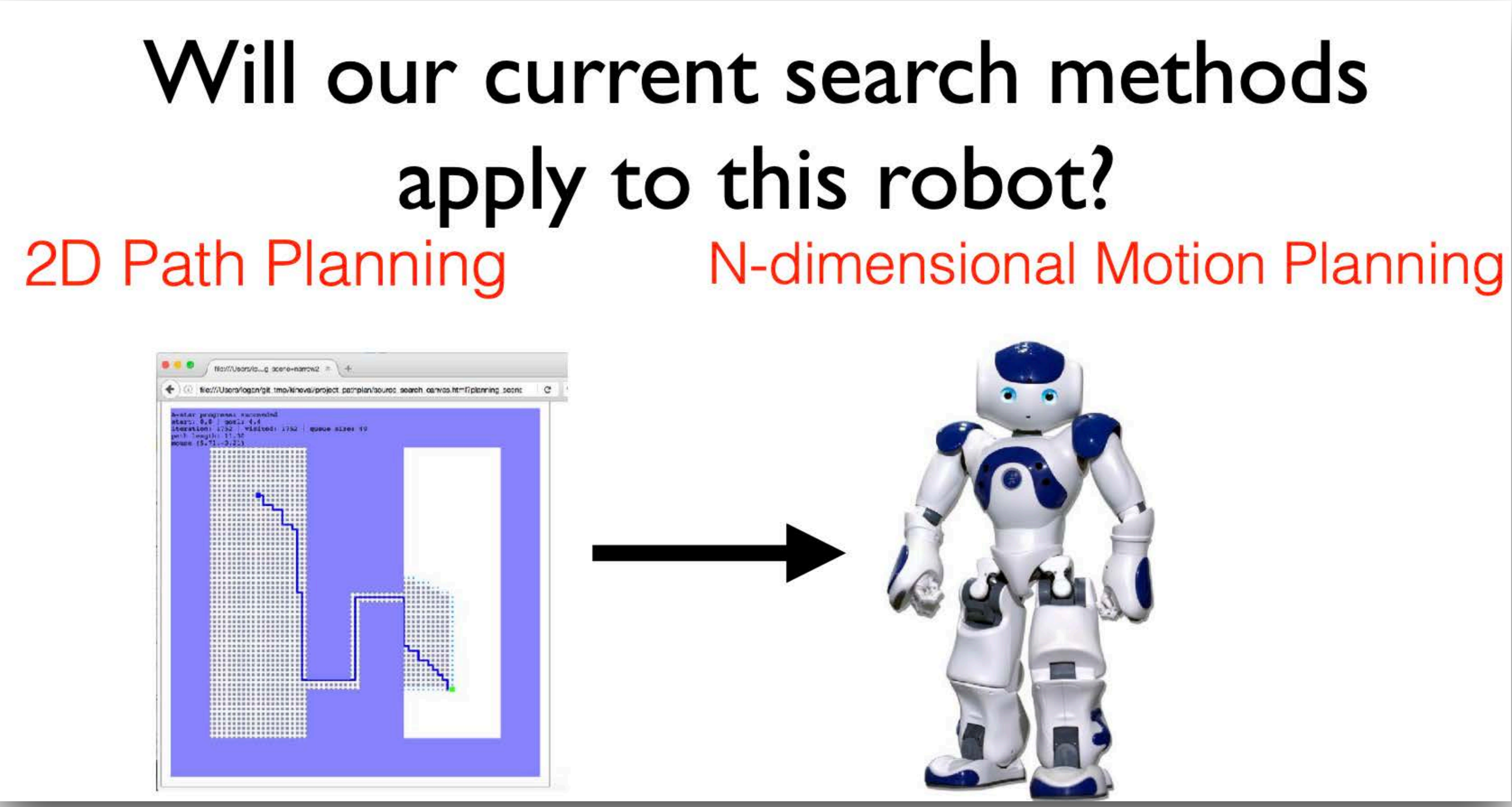
Planning - IV - Sampling-based Planning

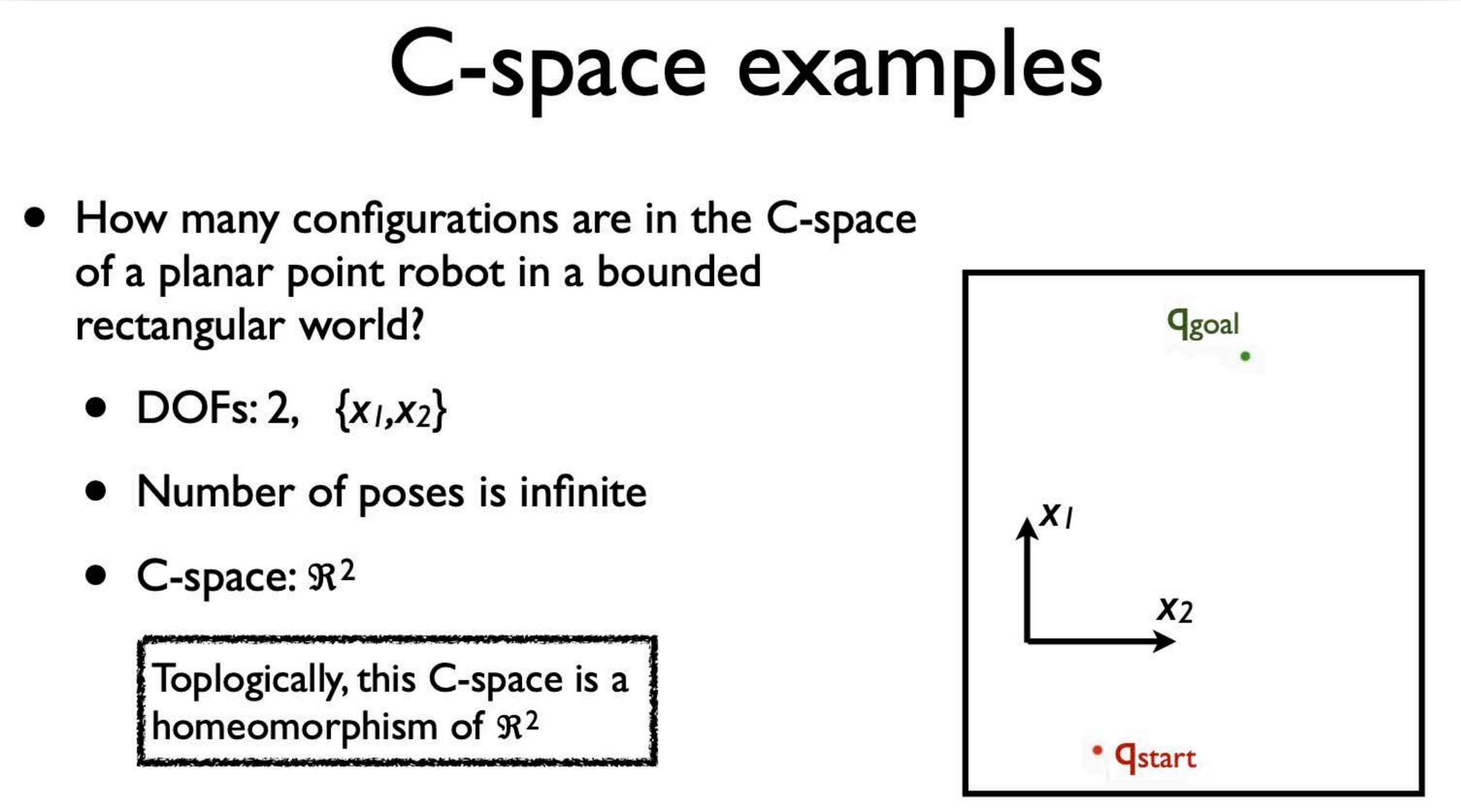


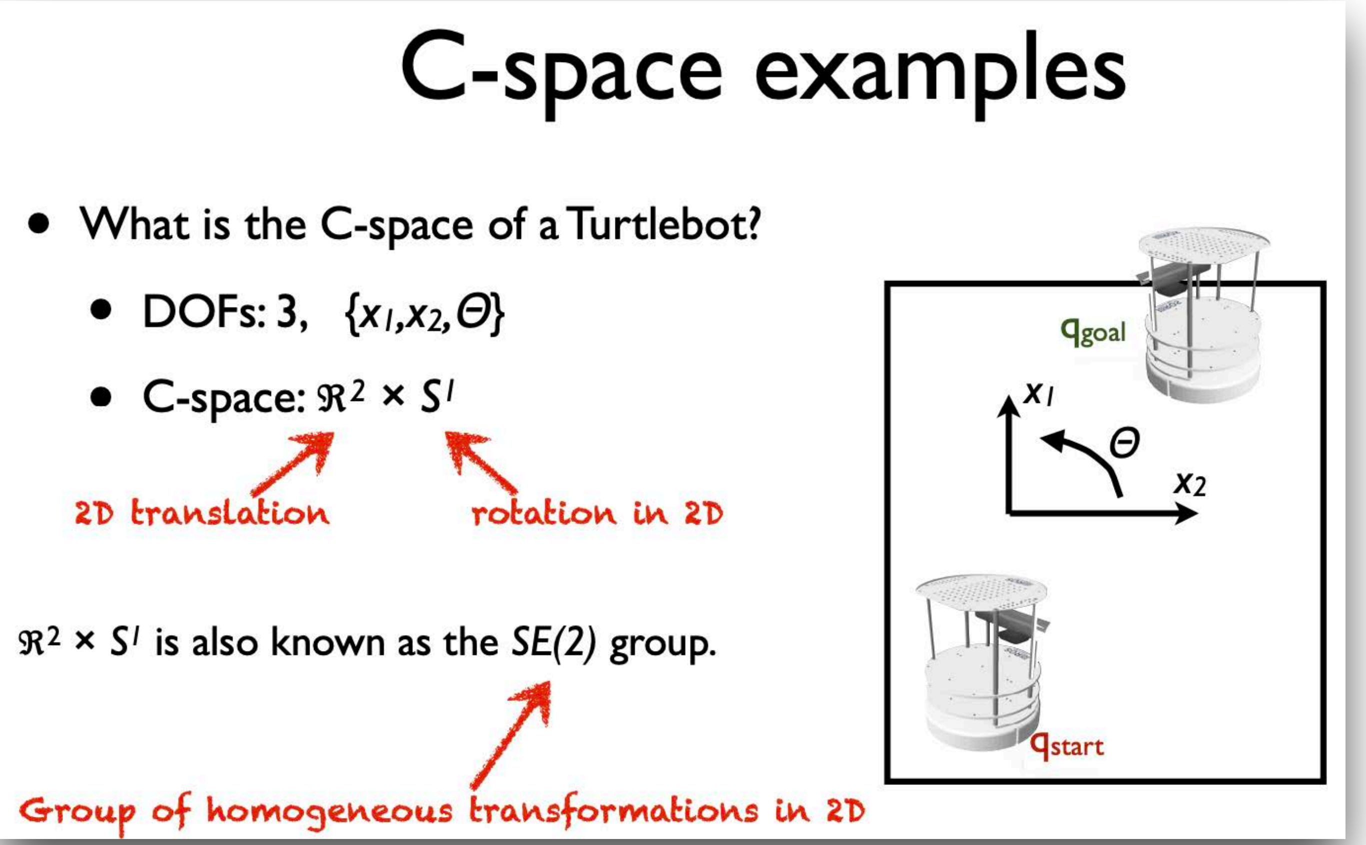
Course Logistics

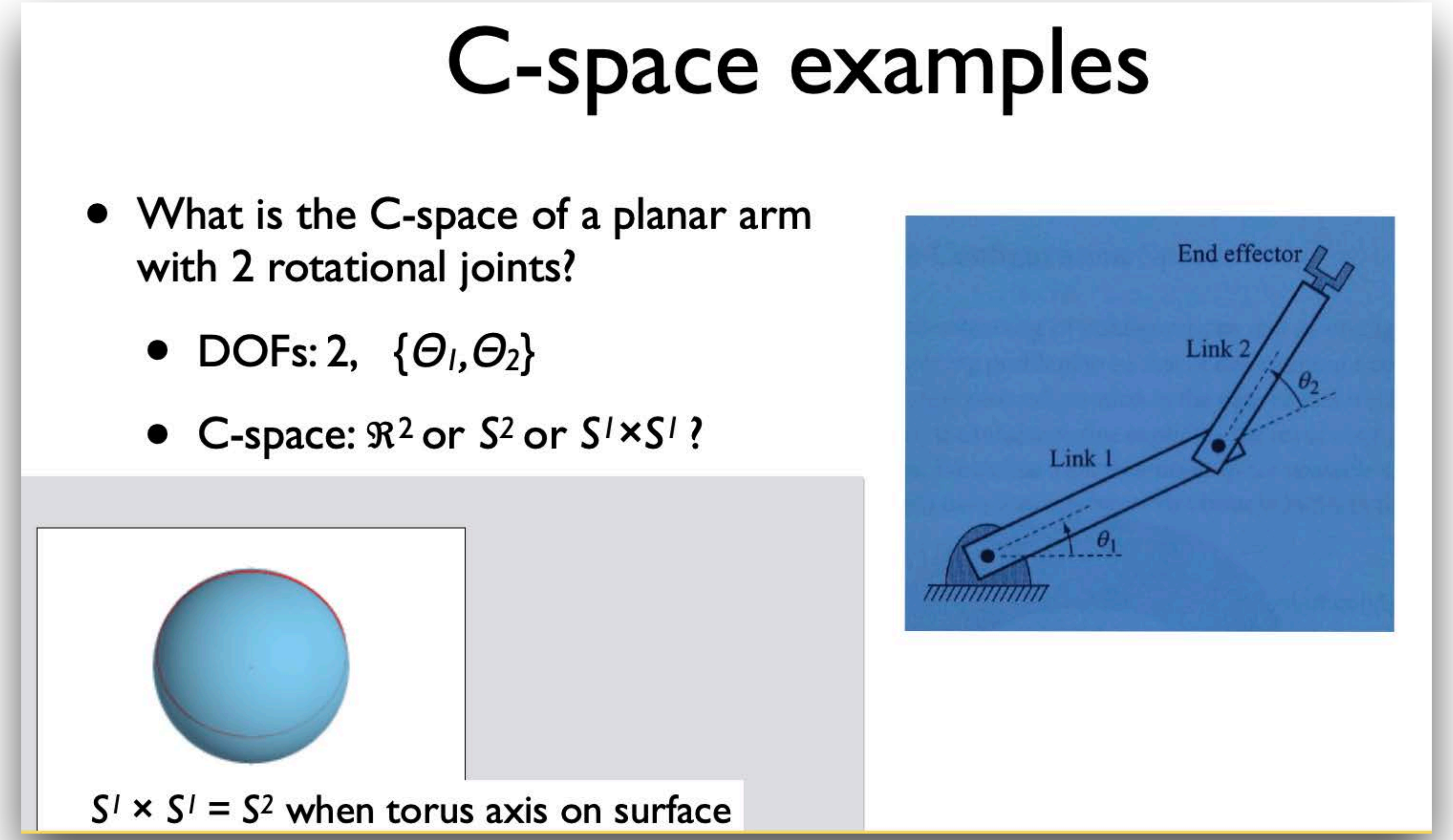
- Quiz 6 will be posted tomorrow at noon and will be due on Wed at noon.
- Project 4 is due on Wed 03/05.
- Project 5 will be posted on 03/05 and will be due on 03/24.

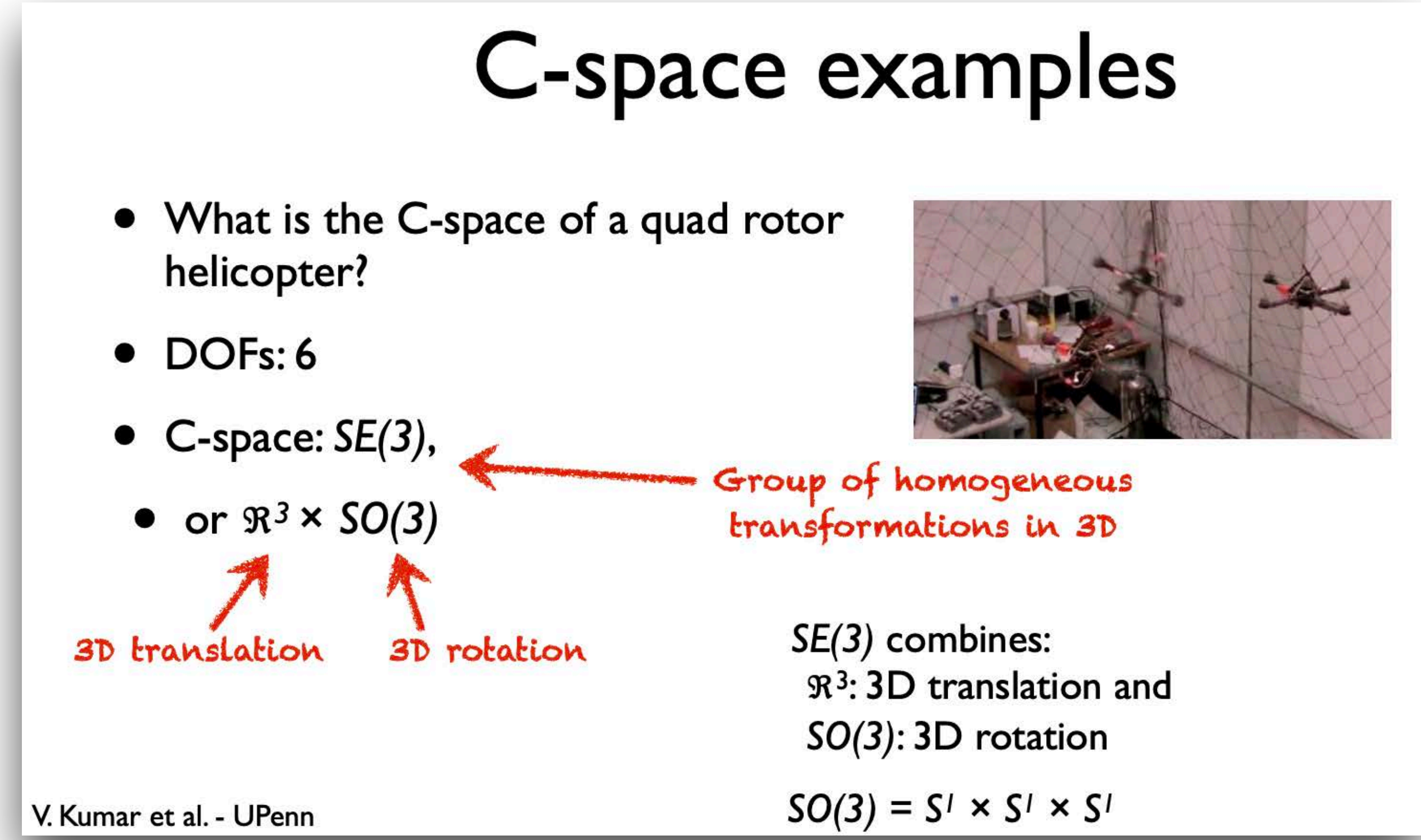
Previously

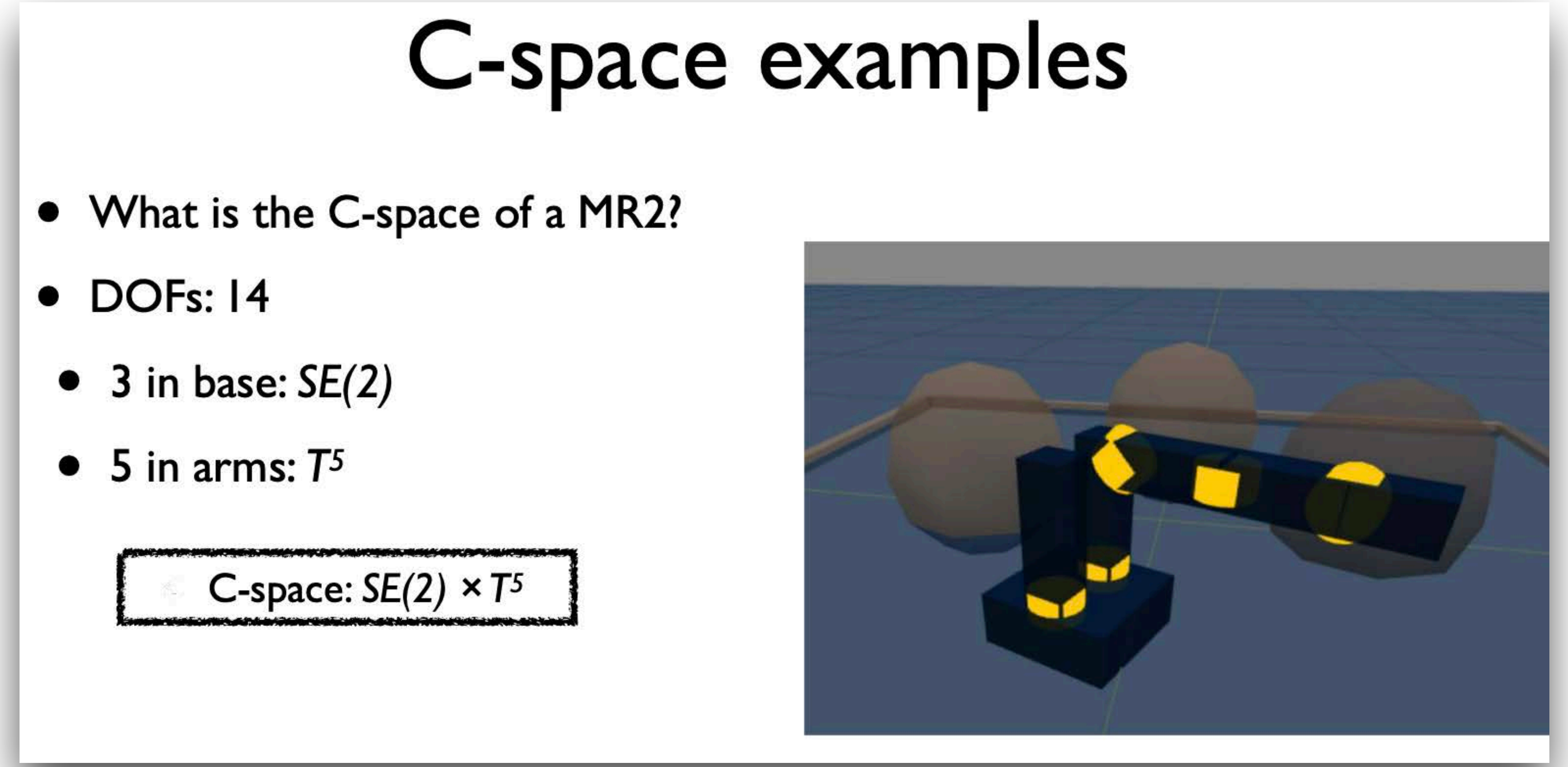

Will our current search methods apply to this robot?
2D Path Planning
N-dimensional Motion Planning


C-space examples
How many configurations are in the C-space of a planar point robot in a bounded rectangular world?
DOFs: 2, $\{x_1,x_2\}$
Number of poses is infinite
C-space: $\Re^2$
Toplogically, this C-space is a homeomorphism of $\Re^2$
q_{goal}
x_1
x_2
q_{start}


C-space examples
What is the C-space of a Turtlebot?
DOFs: 3, $\{x_1,x_2,\Theta\}$
C-space: $\Re^2 \times S^1$
2D translation
rotation in 2D
$\Re^2 \times S^1$ is also known as the $SE(2)$ group.
Group of homogeneous transformations in 2D
q_{goal}
x_1
Θ
x_2
q_{start}


C-space examples
What is the C-space of a planar arm with 2 rotational joints?
DOFs: 2, $\{\Theta_1,\Theta_2\}$
C-space: $\Re^2$ or S^2 or $S^1 \times S^1$?
$S^1 \times S^1 = S^2$ when torus axis on surface
End effector
Link 2
Link 1
θ_2
θ_1

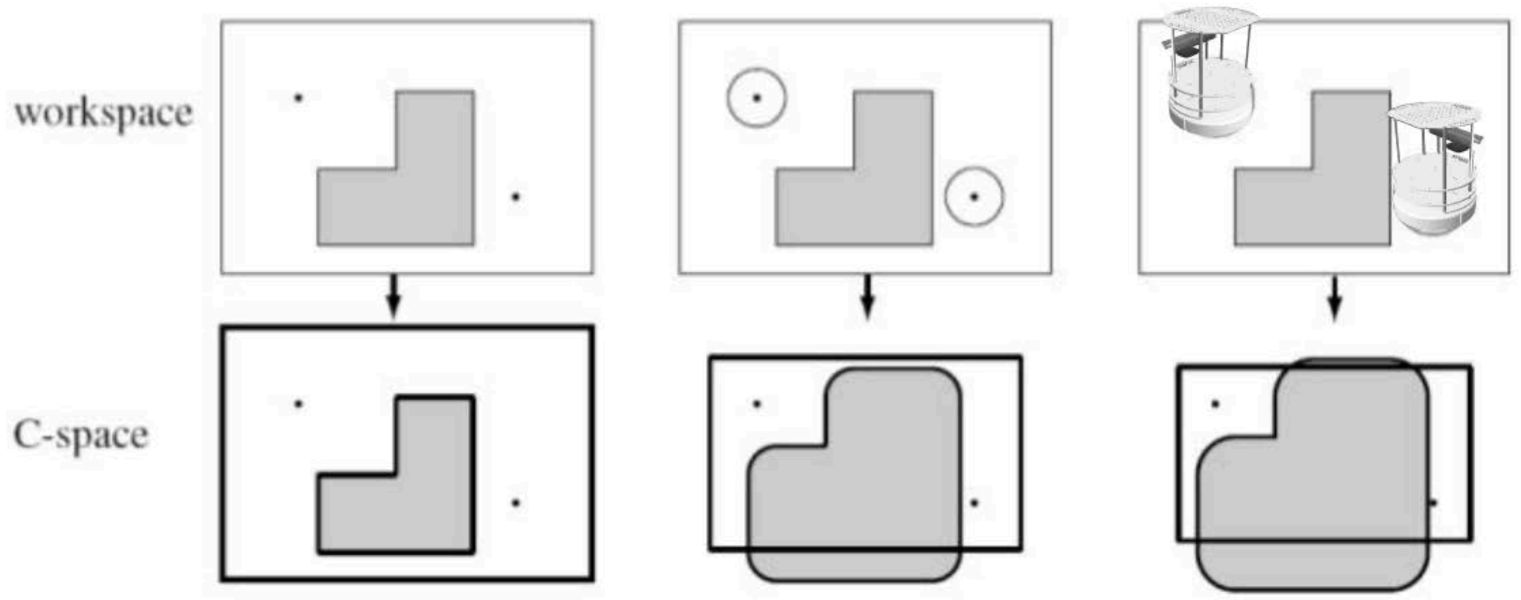

C-space examples
What is the C-space of a quad rotor helicopter?
DOFs: 6
C-space: $SE(3)$,
or $\Re^3 \times SO(3)$
Group of homogeneous transformations in 3D
3D translation
3D rotation
$SE(3)$ combines:
$\Re^3$: 3D translation and
$SO(3)$: 3D rotation
V. Kumar et al. - UPenn
$SO(3) = S^1 \times S^1 \times S^1$


C-space examples
What is the C-space of a MR2?
DOFs: 14
3 in base: $SE(2)$
5 in arms: T^5
C-space: $SE(2) \times T^5$

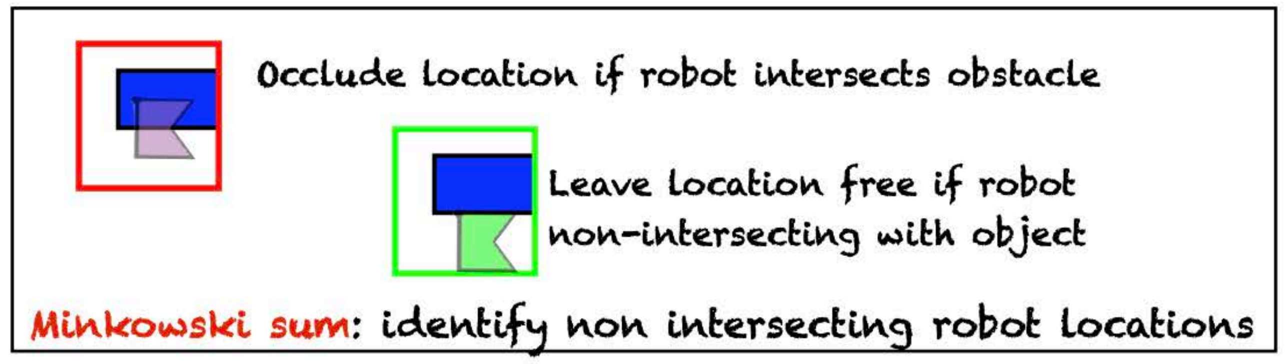
Previously

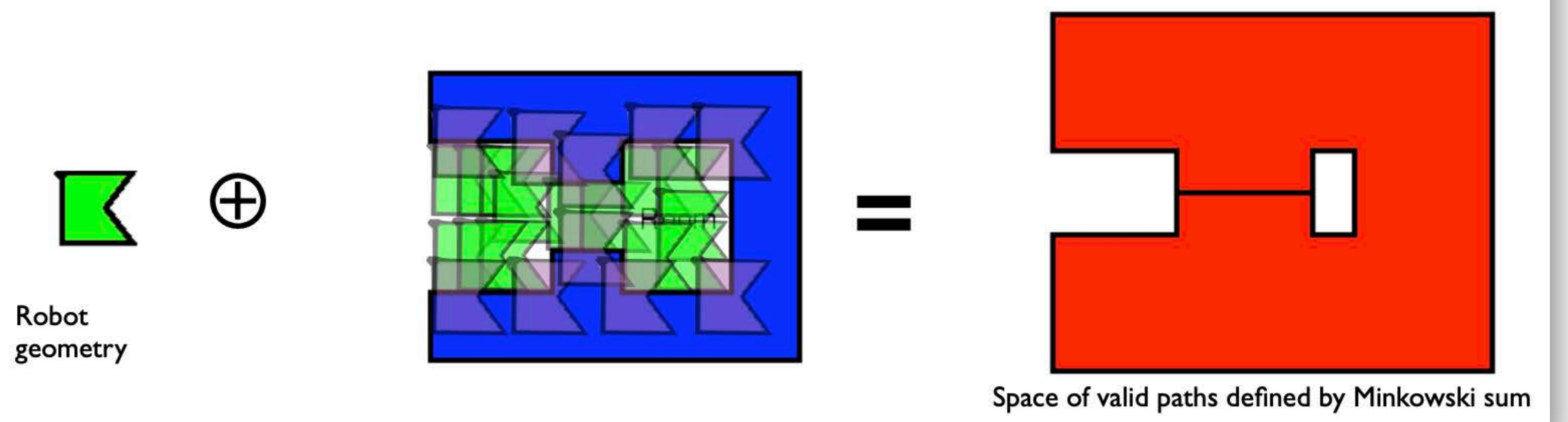
Robot Geometry

- Turtlebot is larger than a point, having a circular radius in the robot's planar workspace
- As this radius increases, the C-space shrinks

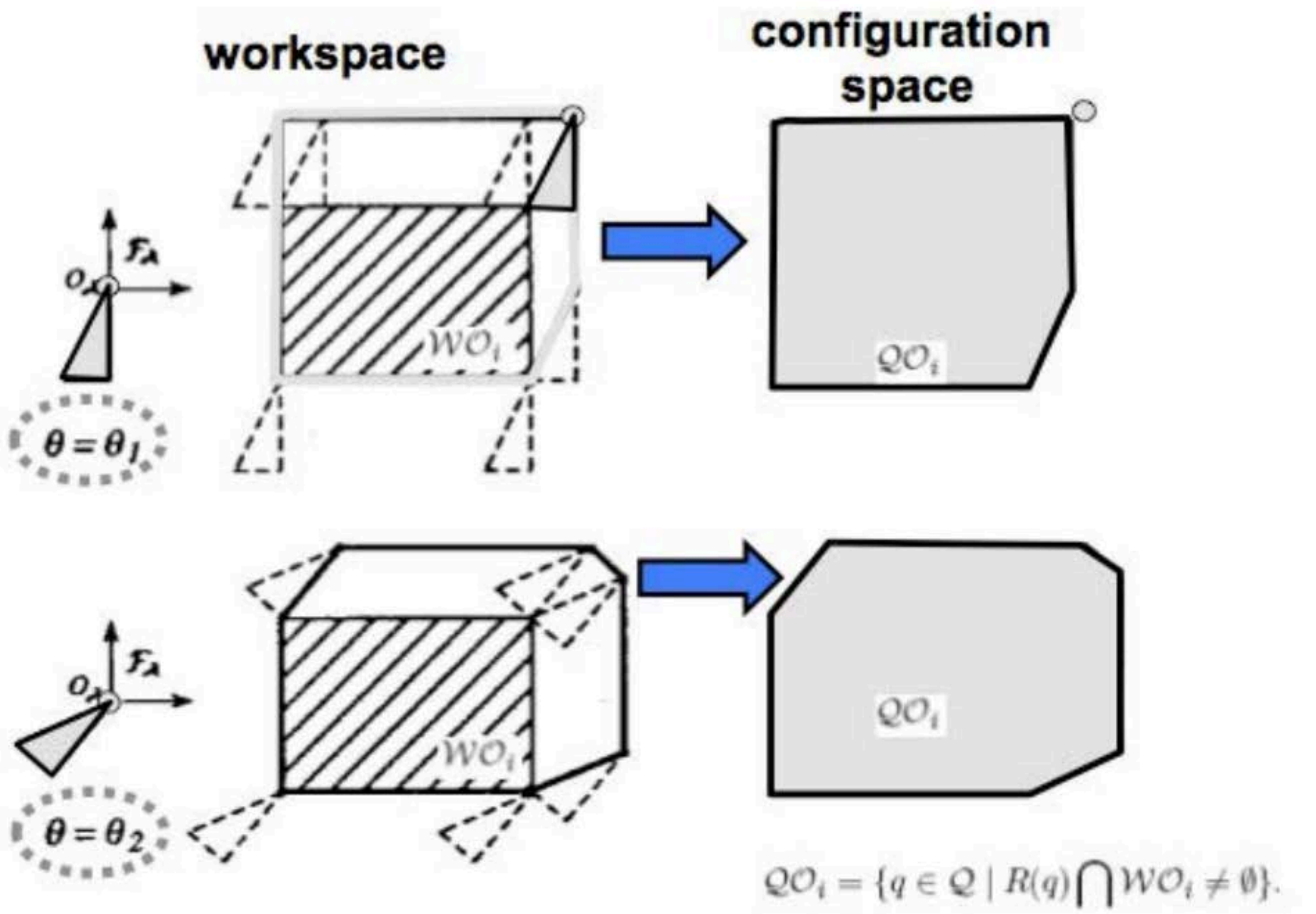


Minkowski Planning

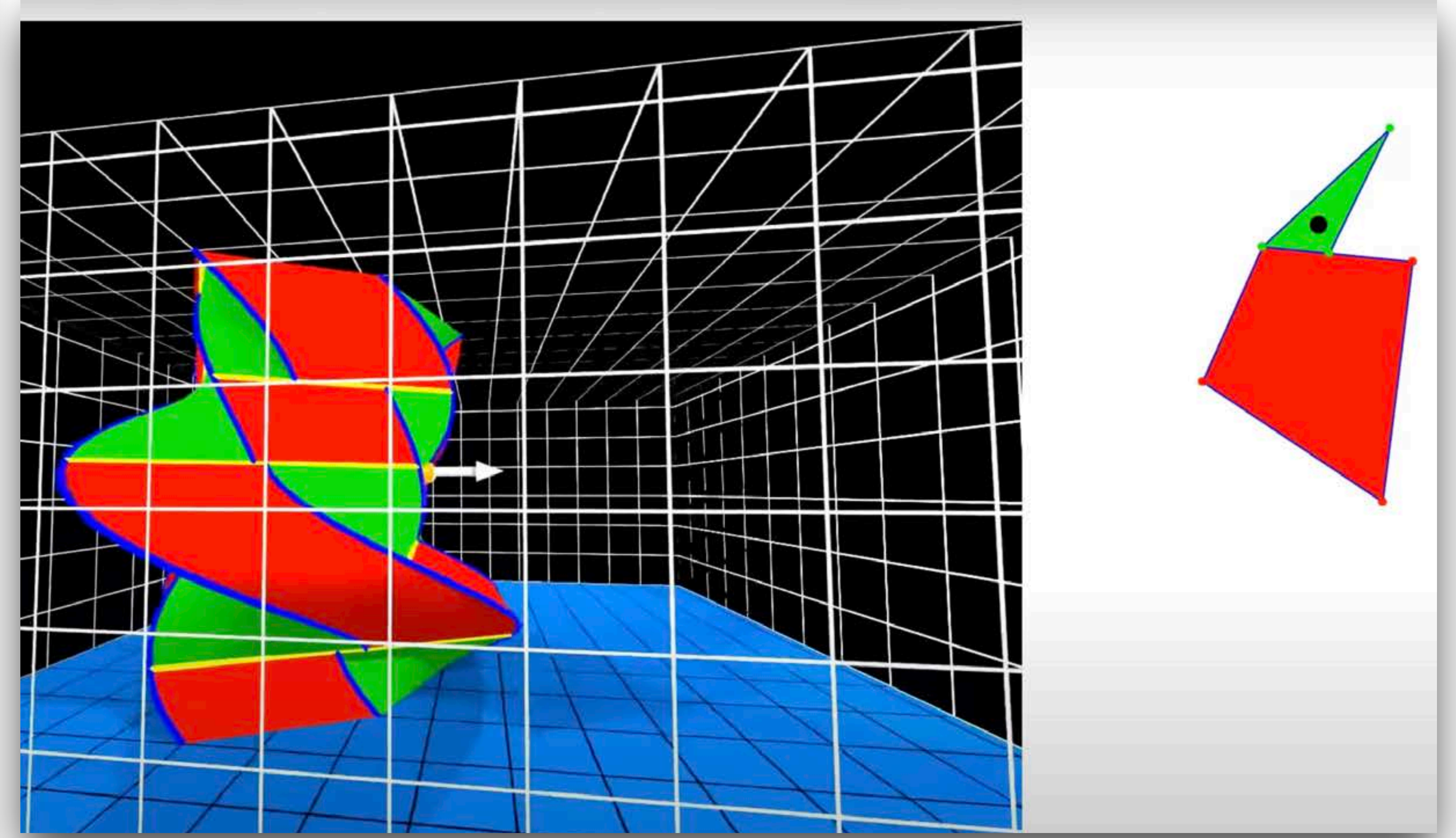


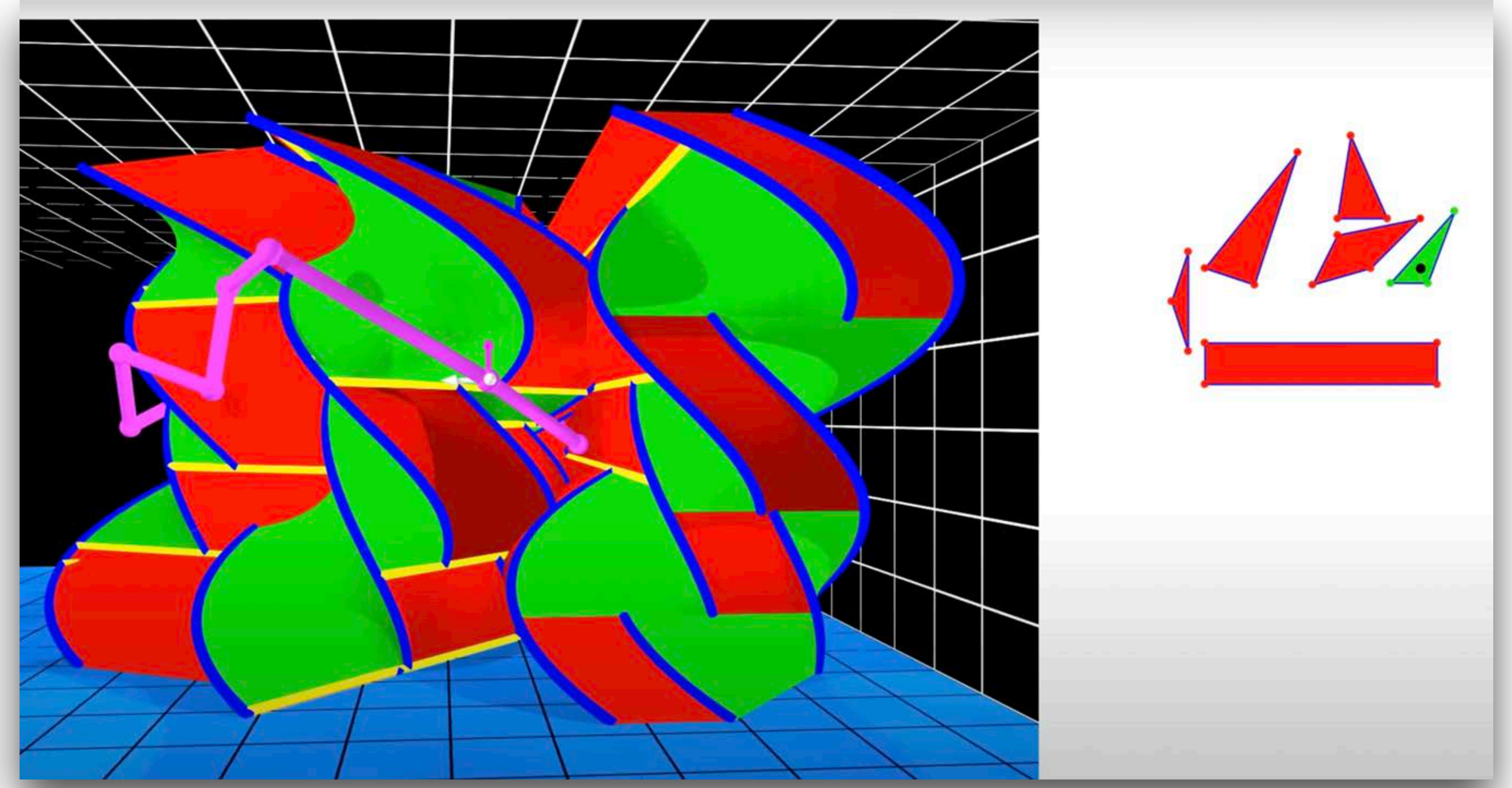


C-space depends on rotation



Robot is a point in the C-space!



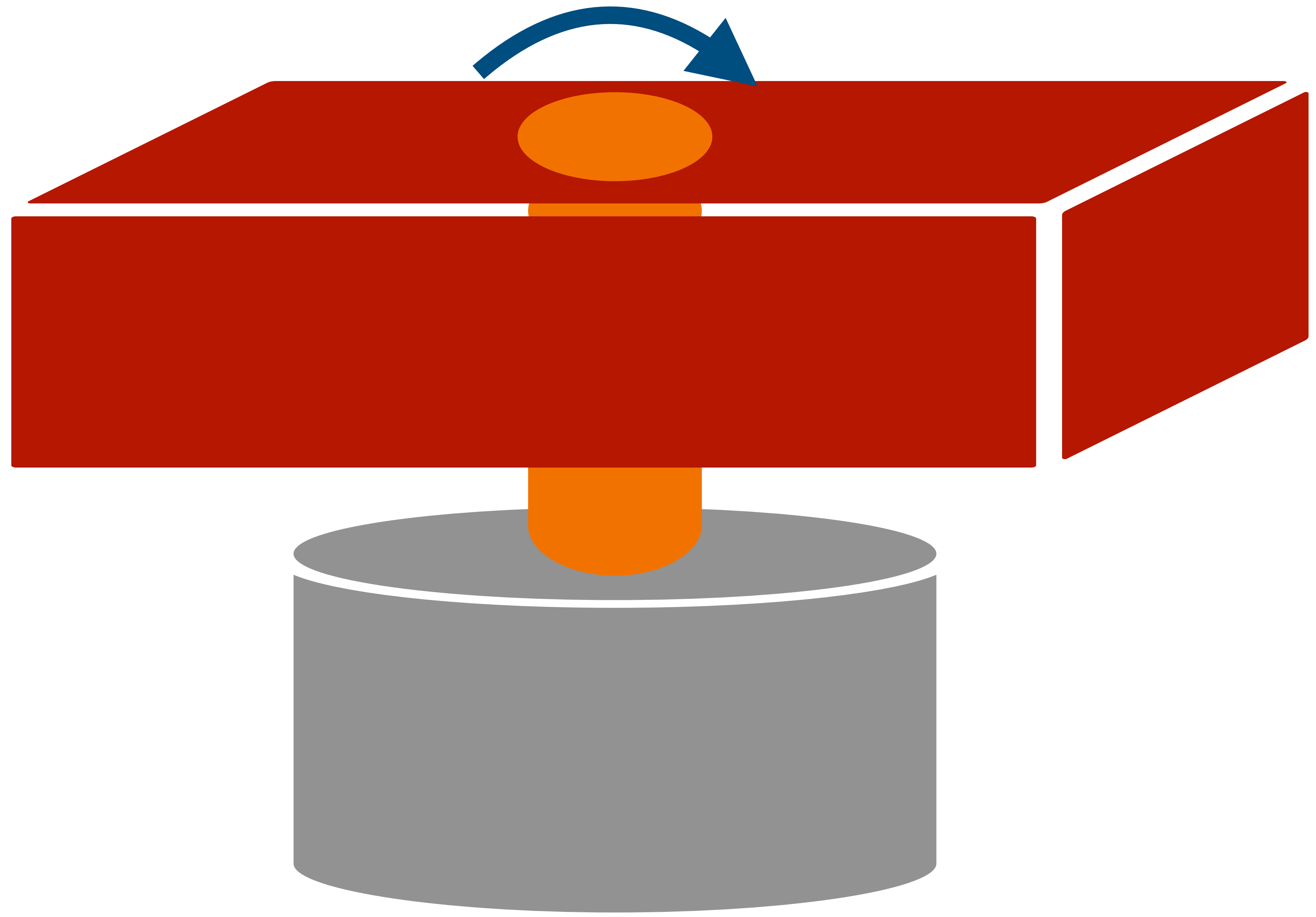


DOF formally:

$$\text{dof} = \sum \text{freedoms of rigid bodies} \; - \# \text{ of independent constraints}$$

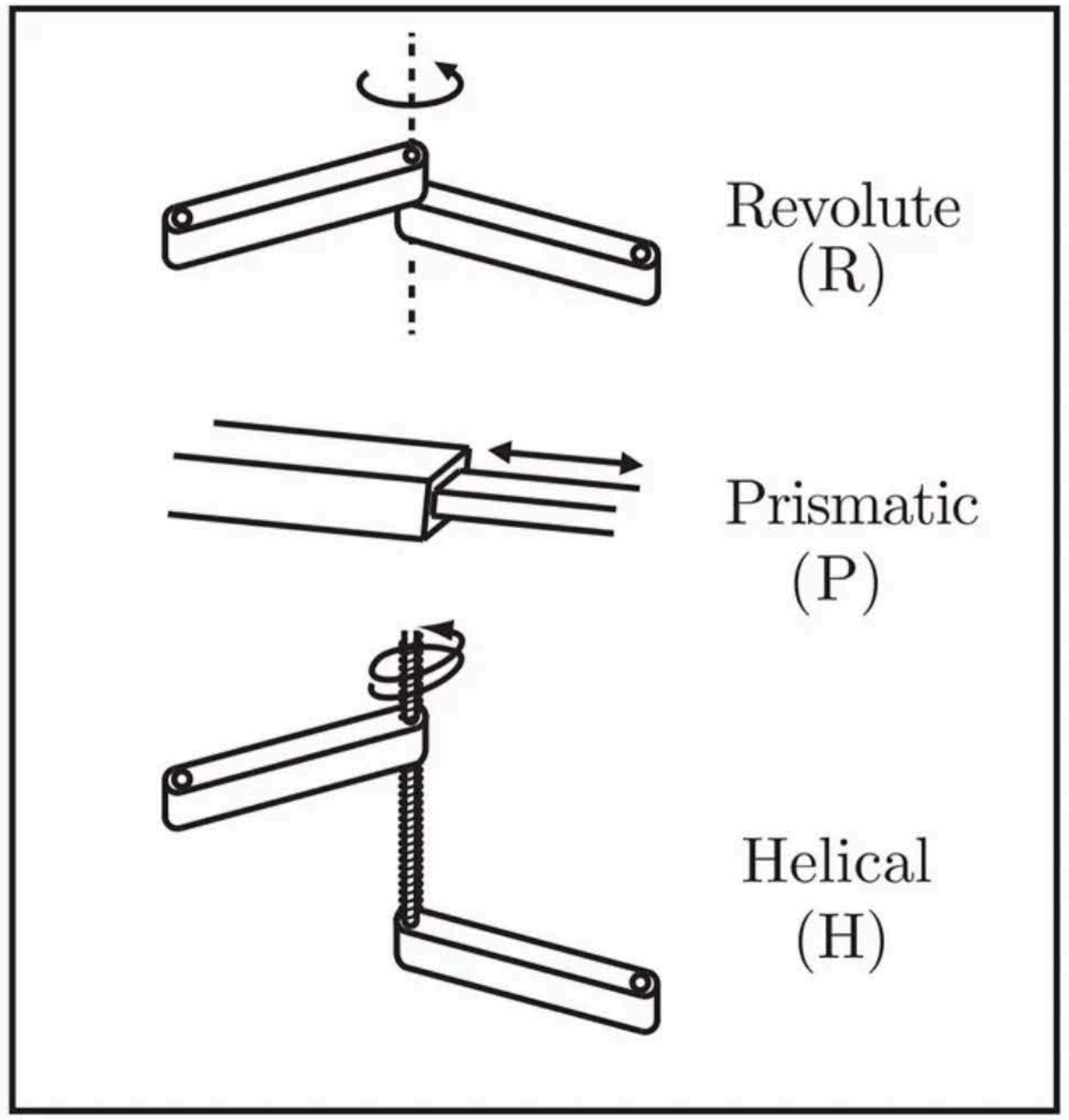
often comes from joints

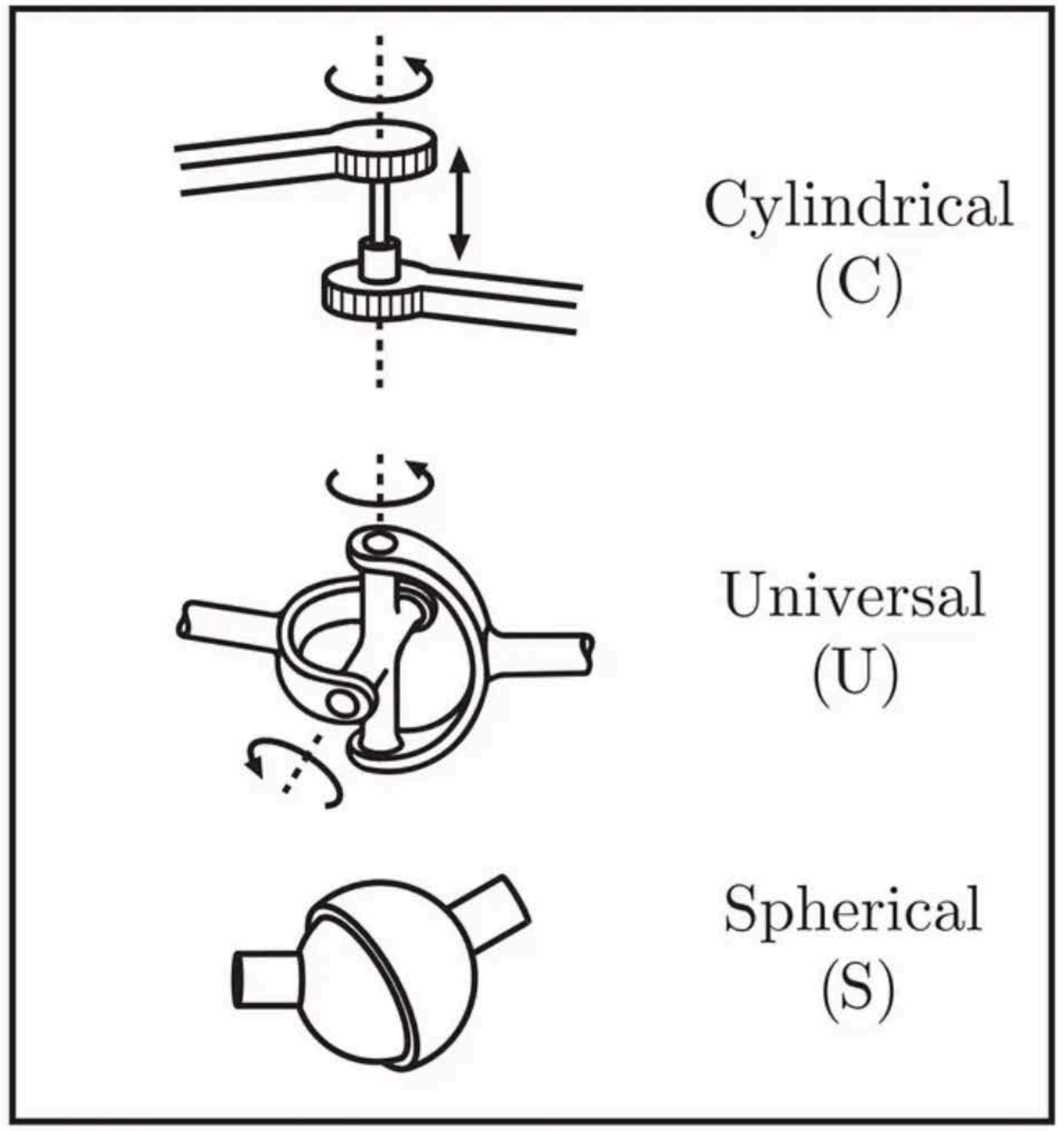
What is the degrees of freedom for this rigid body?



If we add a revolute joint, what happens to the degrees of freedom of this system?

Joints & Constraints





Joint type	dof f
Revolute (R)	1
Prismatic (P)	1
Helical (H)	1
Cylindrical (C)	2
Universal (U)	2
Spherical (S)	3

From the book MODERN ROBOTICS
by Kevin M. Lynch and Frank C. Park May 3, 2017

DOF formally

$$\text{dof} = \sum \text{freedoms of rigid bodies} - \#\text{ of independent constraints}$$

$N = \#$ of bodies, including the ground

$J = \#$ of joints

$m = 6$ for spatial bodies; 3 for planar bodies

$$\text{dof} = \underbrace{m(N-1)}_{\text{Rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{Joint Constraints}}$$

$$\text{dof} = m(N-1) - \sum_{i=1}^{J} (m - f_i)$$

Only applicable if the constraints provided by the joints are independent →

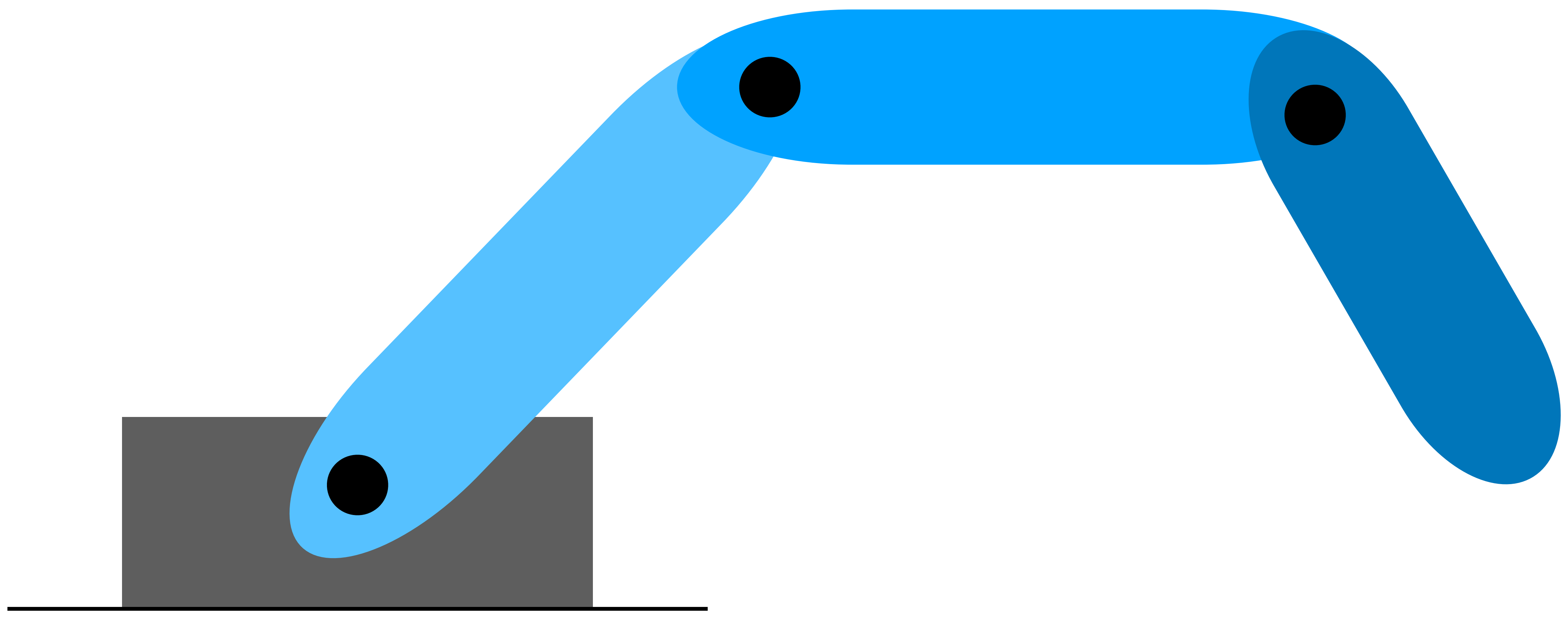
$$\text{dof} = m(N - J - 1) + \sum_{i=1}^{J} (f_i)$$

Grübler's formula

DOF formally: Example 1

$$\text{dof} = m(N - J - 1) + \sum_{i=1}^{J} (f_i)$$

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
Cylindrical (C)	2	N/A	4
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3



3R Serial “open-chain” Robot

$m =$ [redacted]

$J =$ [redacted]

$N =$ [redacted]

dof = [redacted]

DOF formally: Example 2

$$\text{dof} = m(N - J - 1) + \sum_{i=1}^{J} (f_i)$$

Joint type	dof f	Constraints c between two planar rigid bodies	Constraints c between two spatial rigid bodies
Revolute (R)	1	2	5
Prismatic (P)	1	2	5
Helical (H)	1	N/A	5
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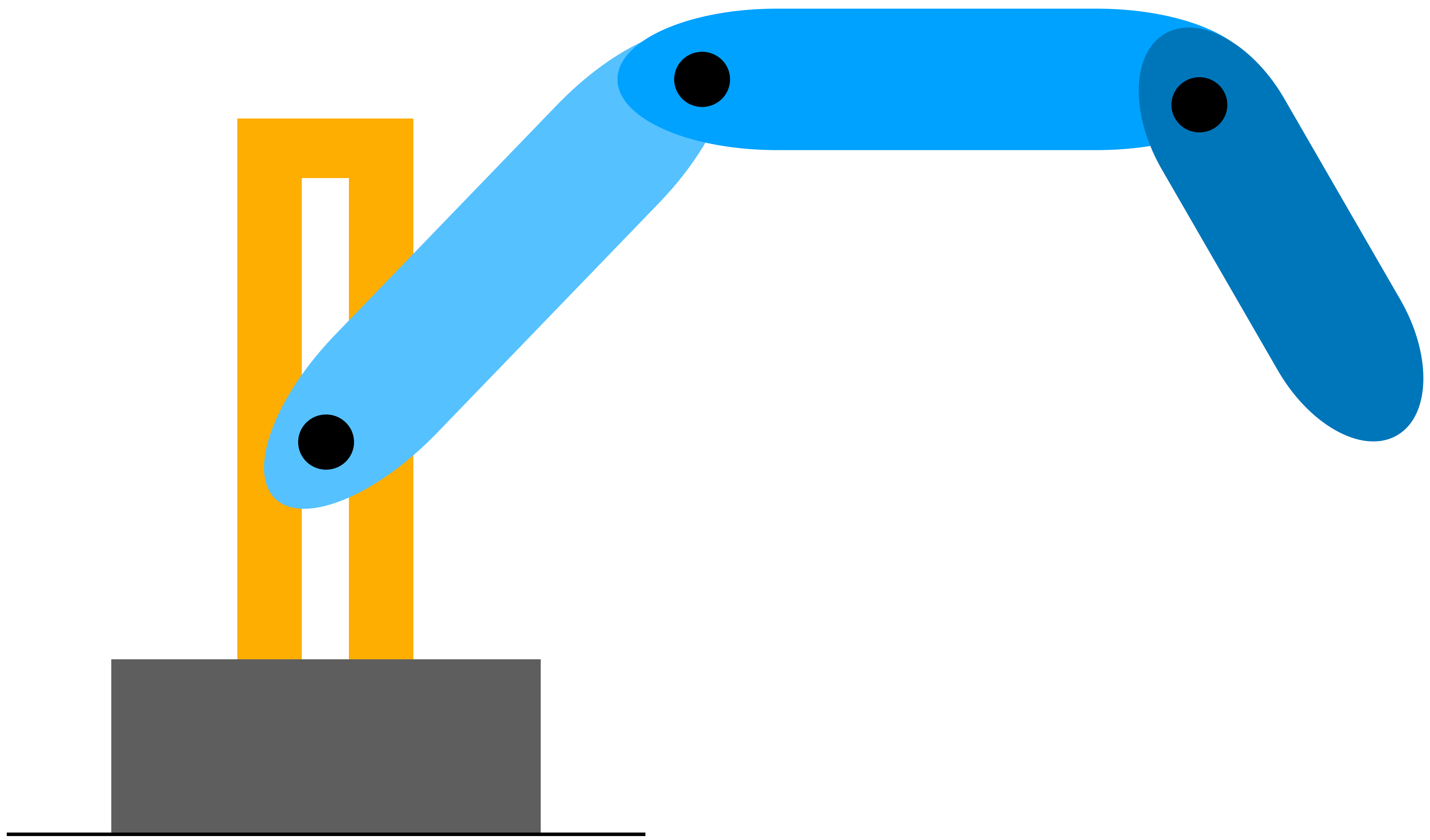
Universal (U)	2	N/A	4
Spherical (S)	3	N/A	3



$$m = 3$$
$$J = 3$$
$$N = 4$$

$$\text{dof} = 3(4 - 3 - 1) + 3 = 3$$

DOF formally

$$\text{dof} = \sum \text{freedoms of rigid bodies} - \#\text{ of independent constraints}$$

$$\text{dof} = \underbrace{m(N-1)}_{\text{Rigid body freedoms}} - \underbrace{\sum_{i=1}^{J} c_i}_{\text{Joint Constraints}}$$

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$m = 6$ for spatial bodies; 3 for planar bodies

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??? Only applicable if the constraints provided by the joints are independent →

$$\text{dof} = m(N-J-1) + \sum_{i=1}^{J} (f_i)$$

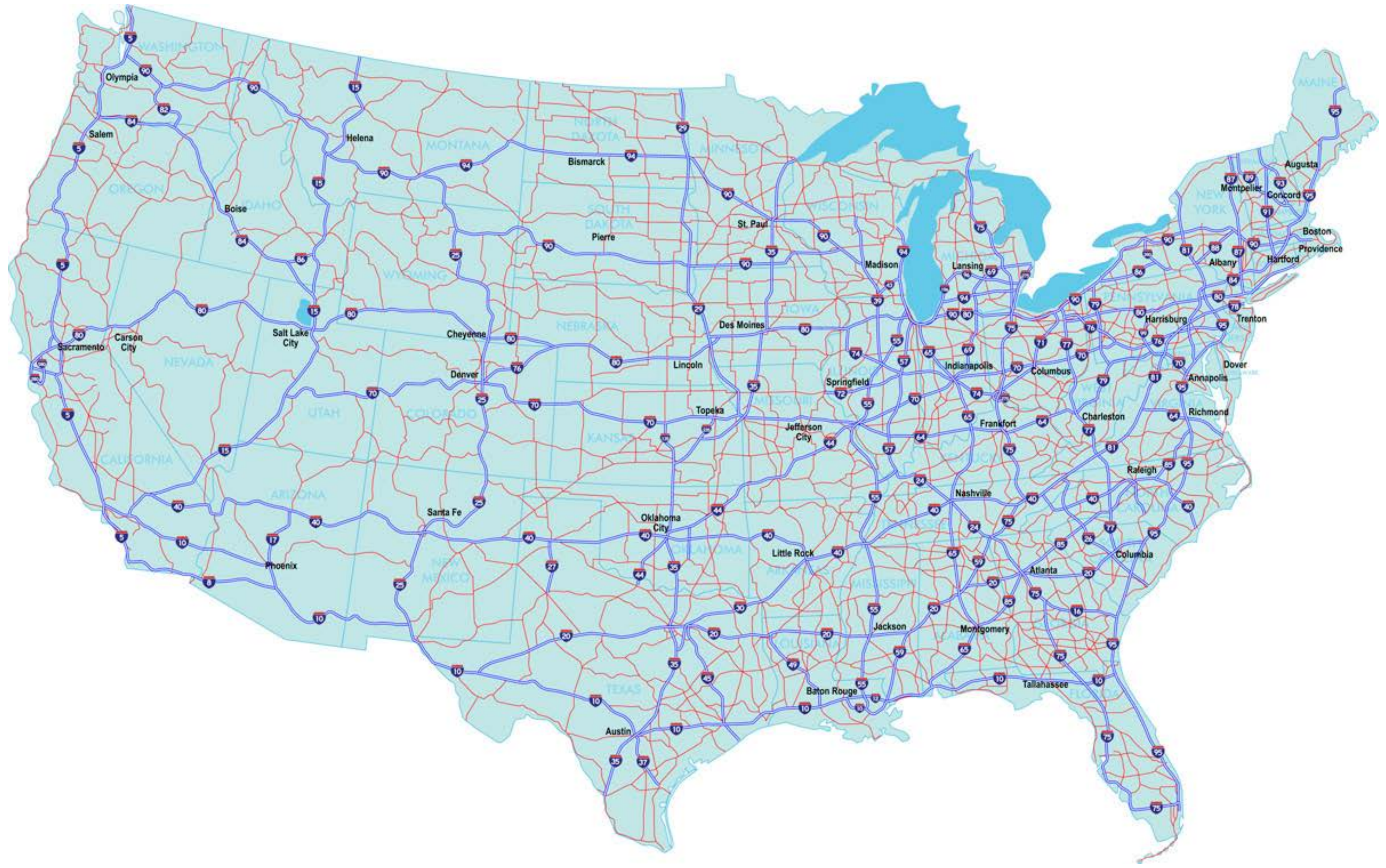
Grübler's formula

Sampling-based Planning

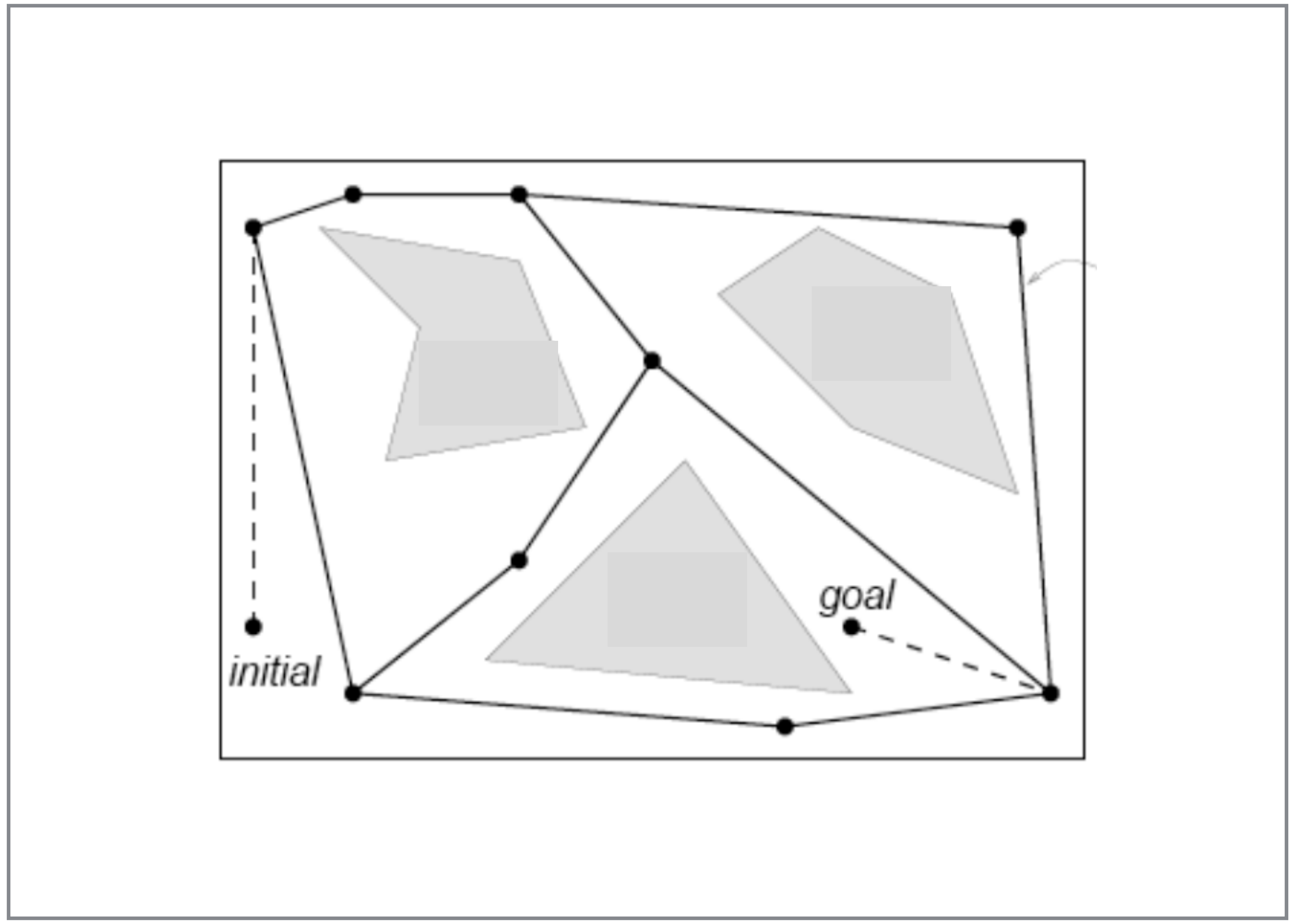
Approaches to motion planning

- Bug algorithms: Bug[0-2], Tangent Bug
- Graph Search (fixed graph)
  - Depth-first, Breadth-first, Dijkstra, A-star, Greedy best-first
- **Sampling-based Search (build graph):**
  - **Probabilistic Road Maps, Rapidly-exploring Random Trees**
- Optimization and local search:
  - Gradient descent, Potential fields, Simulated annealing, Wavefront

Roadmaps

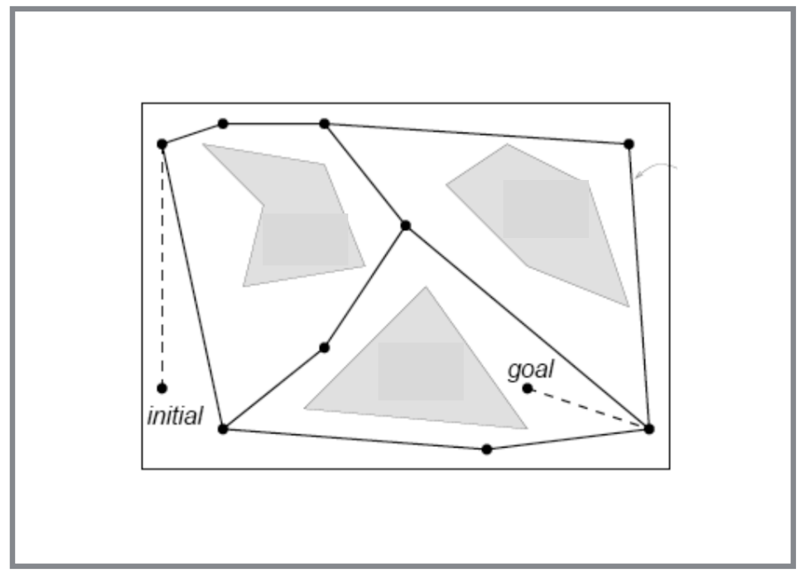


Roadmap over geolocations

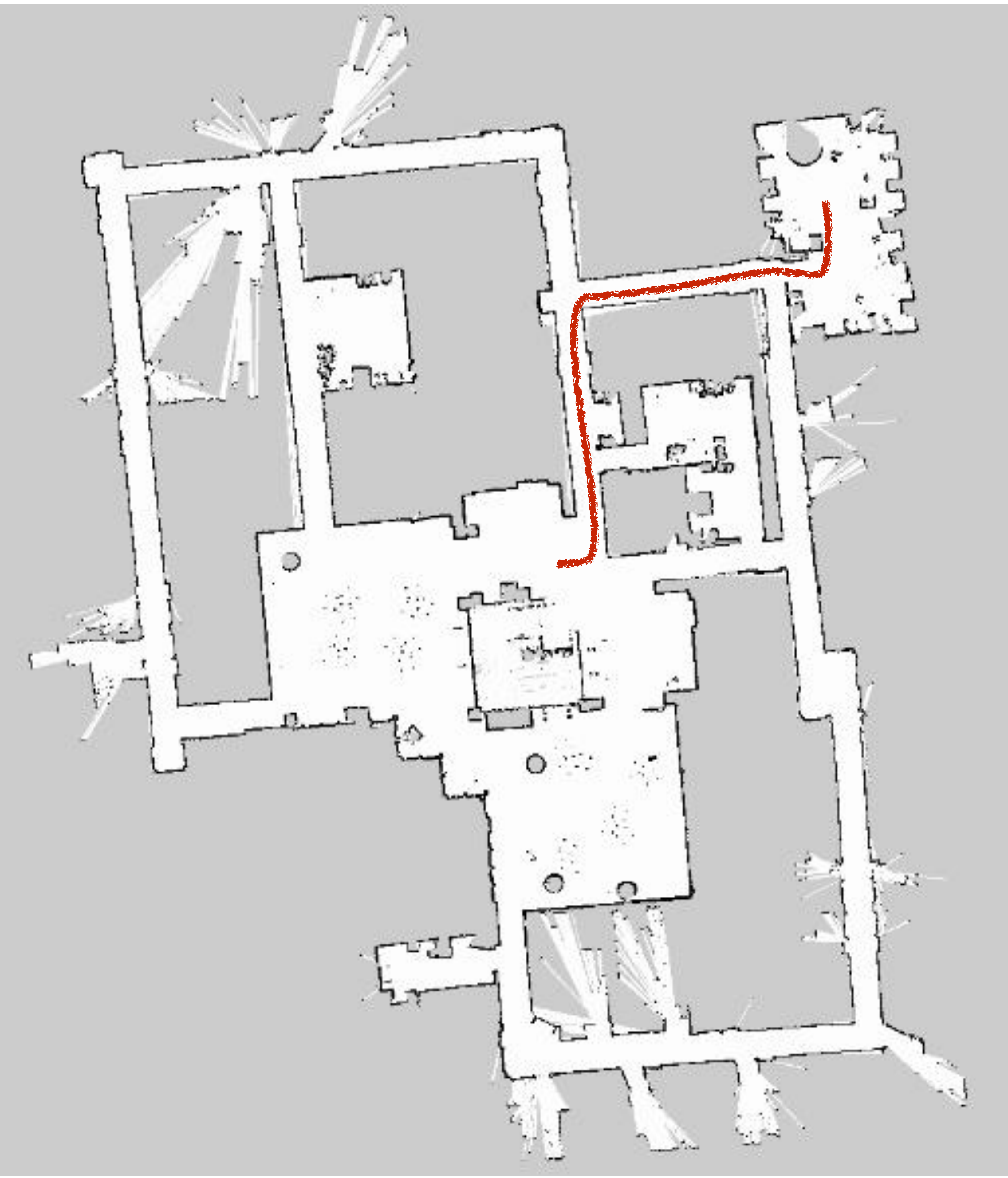


Roadmap over robot configurations

Roadmaps

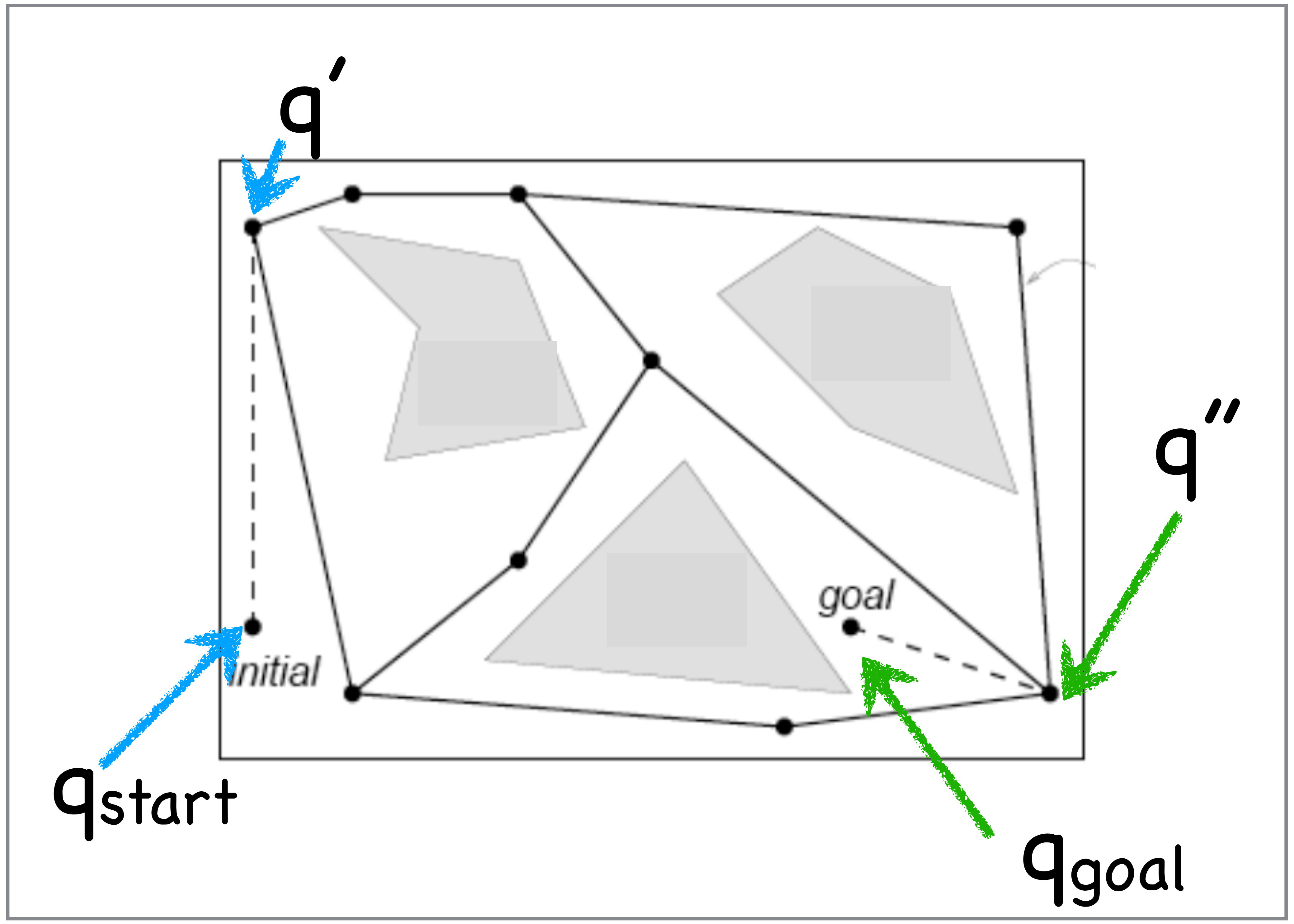


- Graph search assumed C-space as a fixed uniform grid
  - finite set of discretized cells
- How does this scale beyond planar navigation?
  - curse of dimensionality
- Roadmaps are a more general notion of graphs in C-space

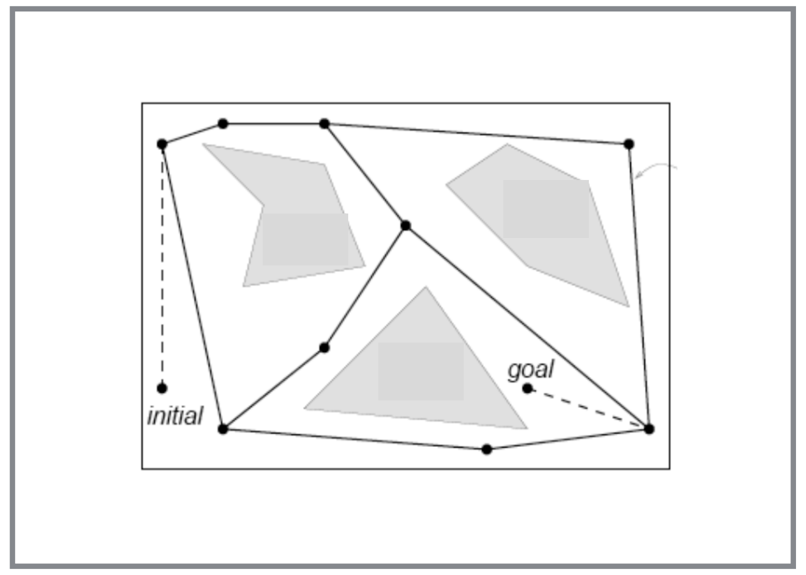


Roadmap Definition

- A roadmap *RM* is a union of curves s.t. all start and goal points in C-space (Q_{free}) can be connected by a path
- Roadmap properties:
  - **Accessibility**: There is a path from $q_{start} \in Q_{free}$ to some $q' \in RM$
  - **Departability**: There is a path from $q'' \in RM$ to $q_{goal} \in Q_{free}$
  - **Connectivity**: there exists a path in *RM* between *q'* and *q''*



Basic Roadmap Planner



1) **Build** the roadmap *RM* as graph *G(V,E)*

   - *V*: nodes are "valid" in C-space in Q_{free}
     - a configuration *q* is valid if it is not in collision and within joint limits
   - *E*: an edge $e(q_1,q_2)$ connects two nodes if a free path connects q_1 and q_2
     - all configurations along edge assumed to be valid

2) **Connect** start and goal configurations to *RM* at *q'* and *q''*, respectively

3) **Find** path in *RM* between *q'* and *q''*

How to build a roadmap?

How to build a roadmap?

2 Approaches

2 Approaches to Roadmaps

Deterministic:

complete algorithms

- Visibility Graph
  - trace lines connecting obstacle polygon vertices
- Voronoi Planning
  - trace edges equidistant from obstacles

Probabilistic:

C-space sampling

- Probabilistic Roadmap (PRM)
  - sample and connect vertices in graph for multiple planning queries
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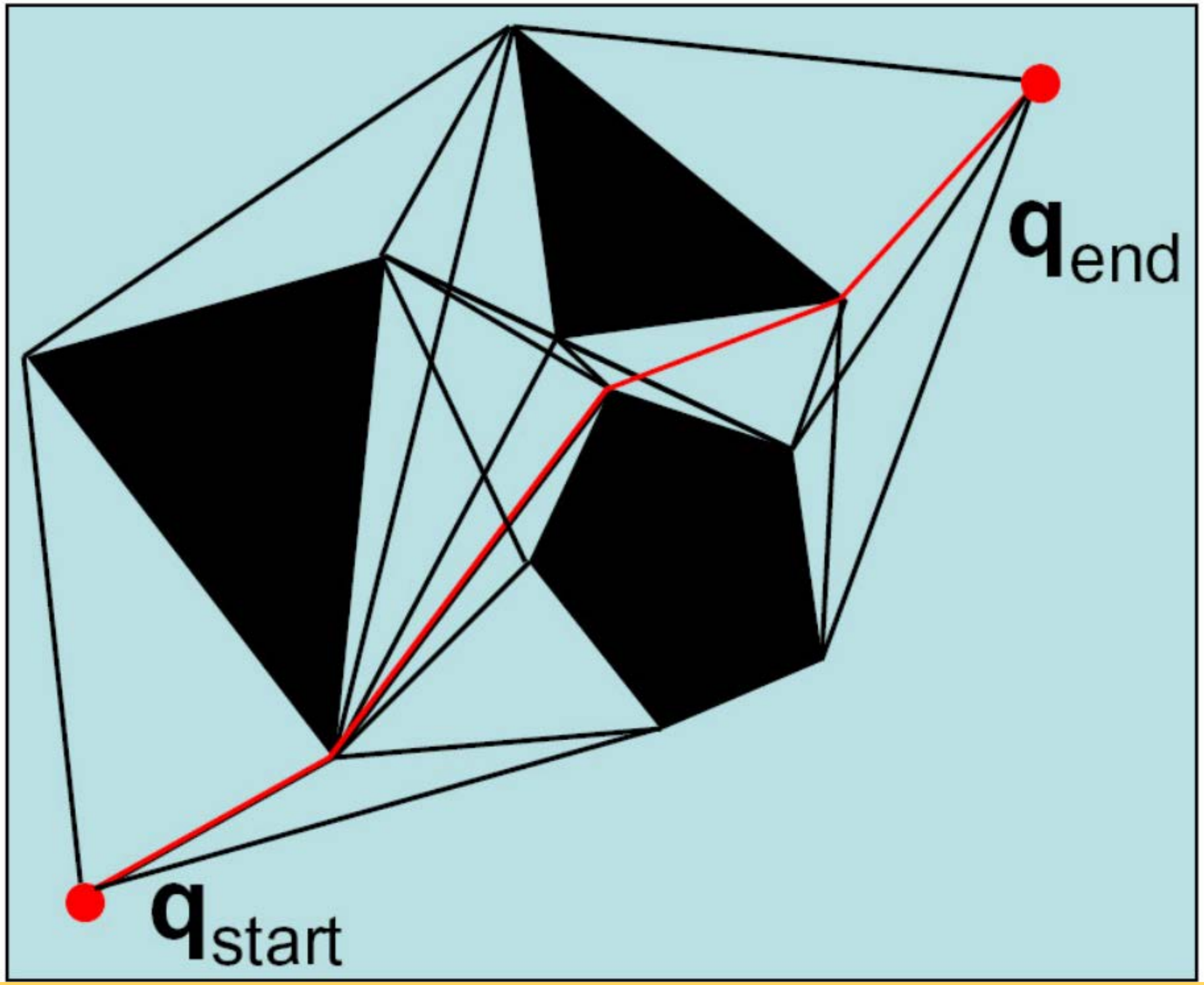

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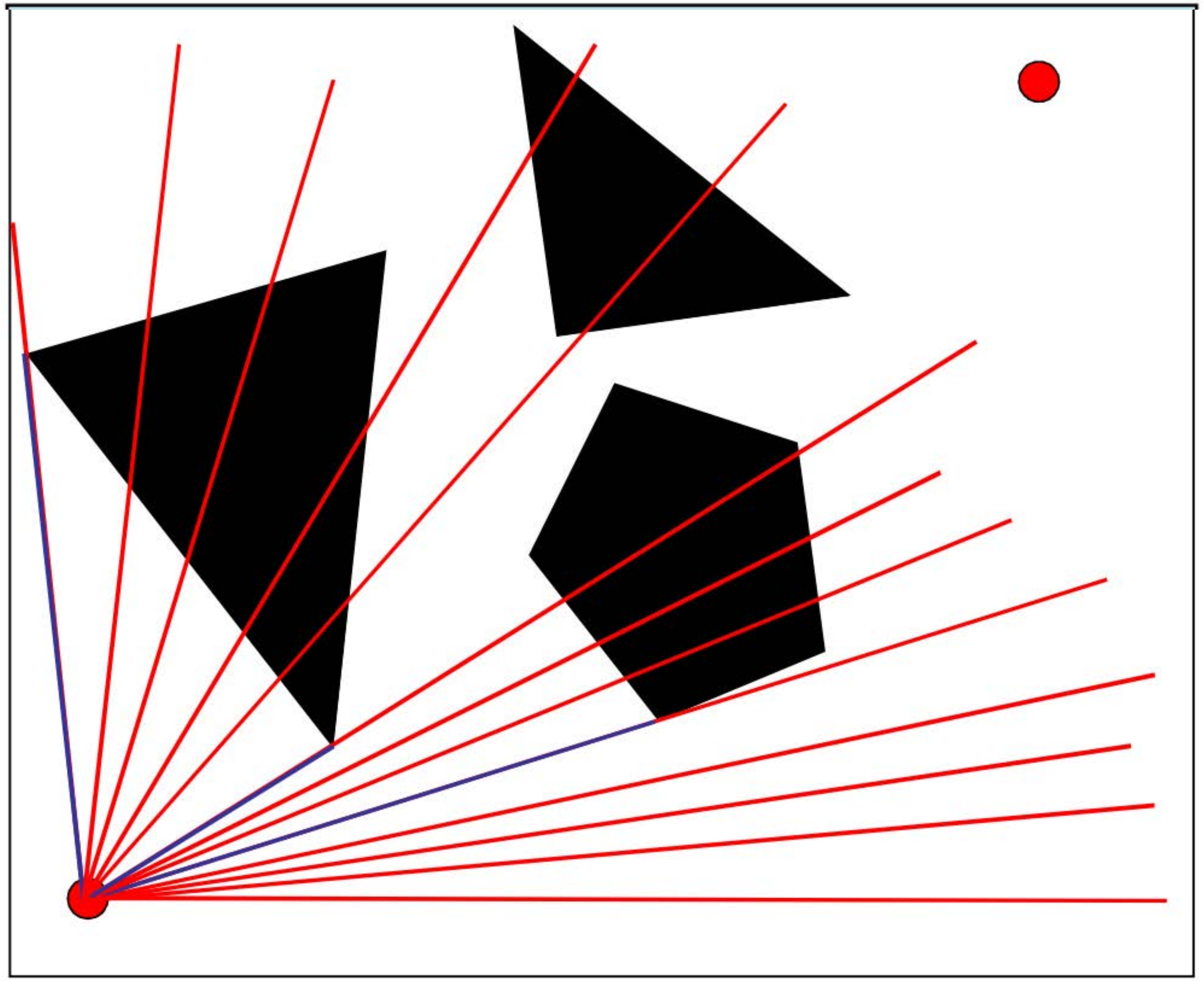

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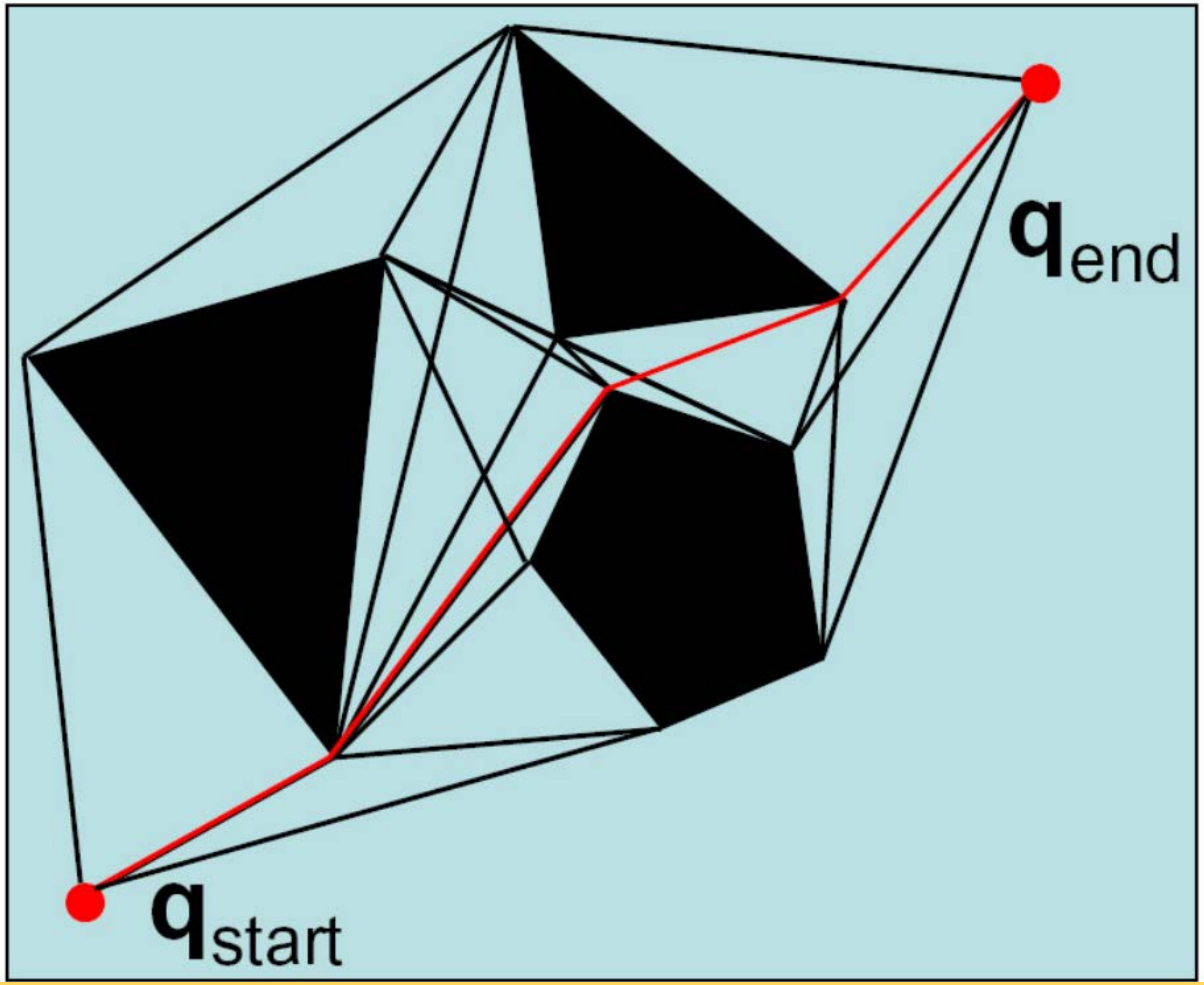

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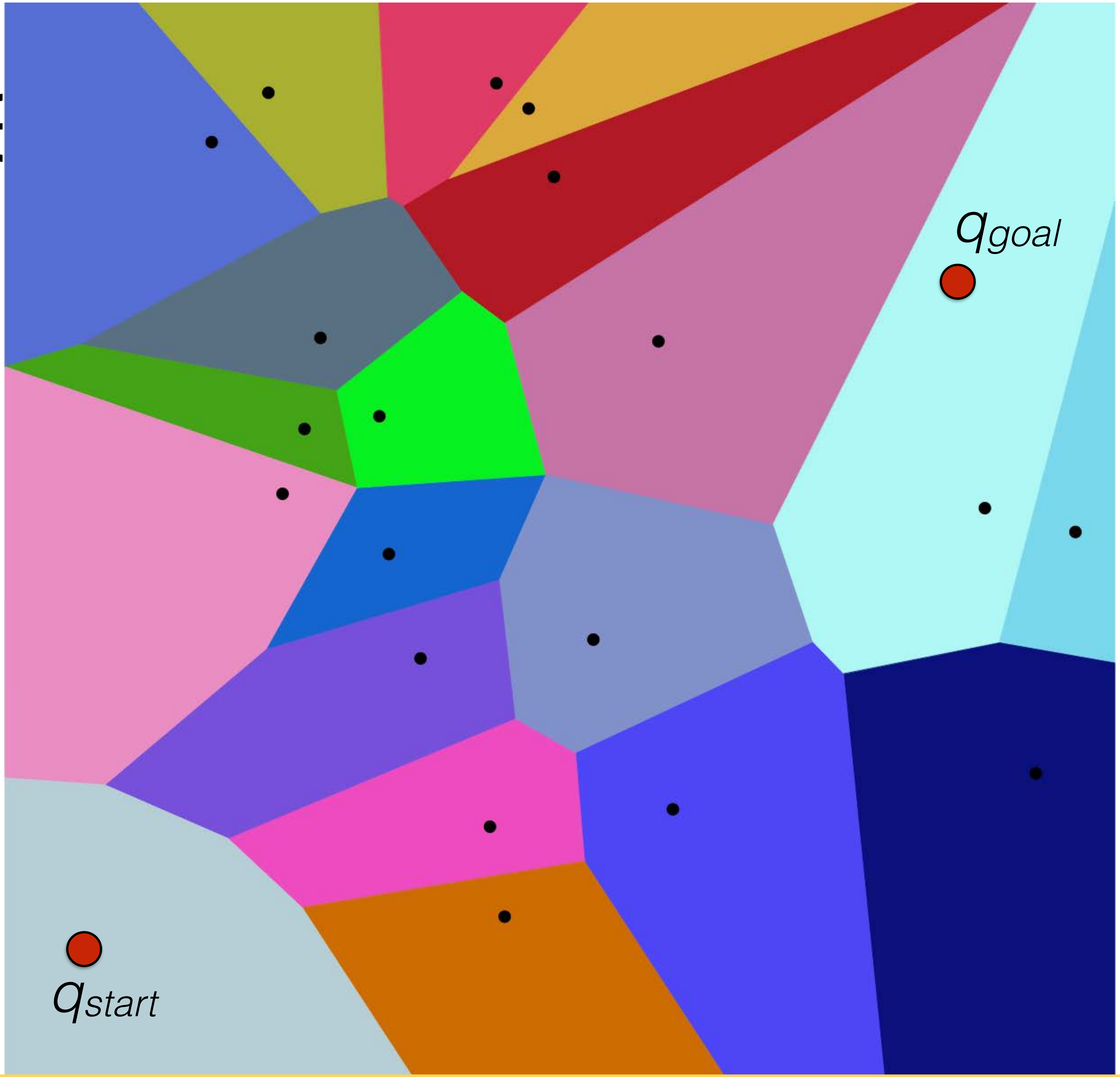

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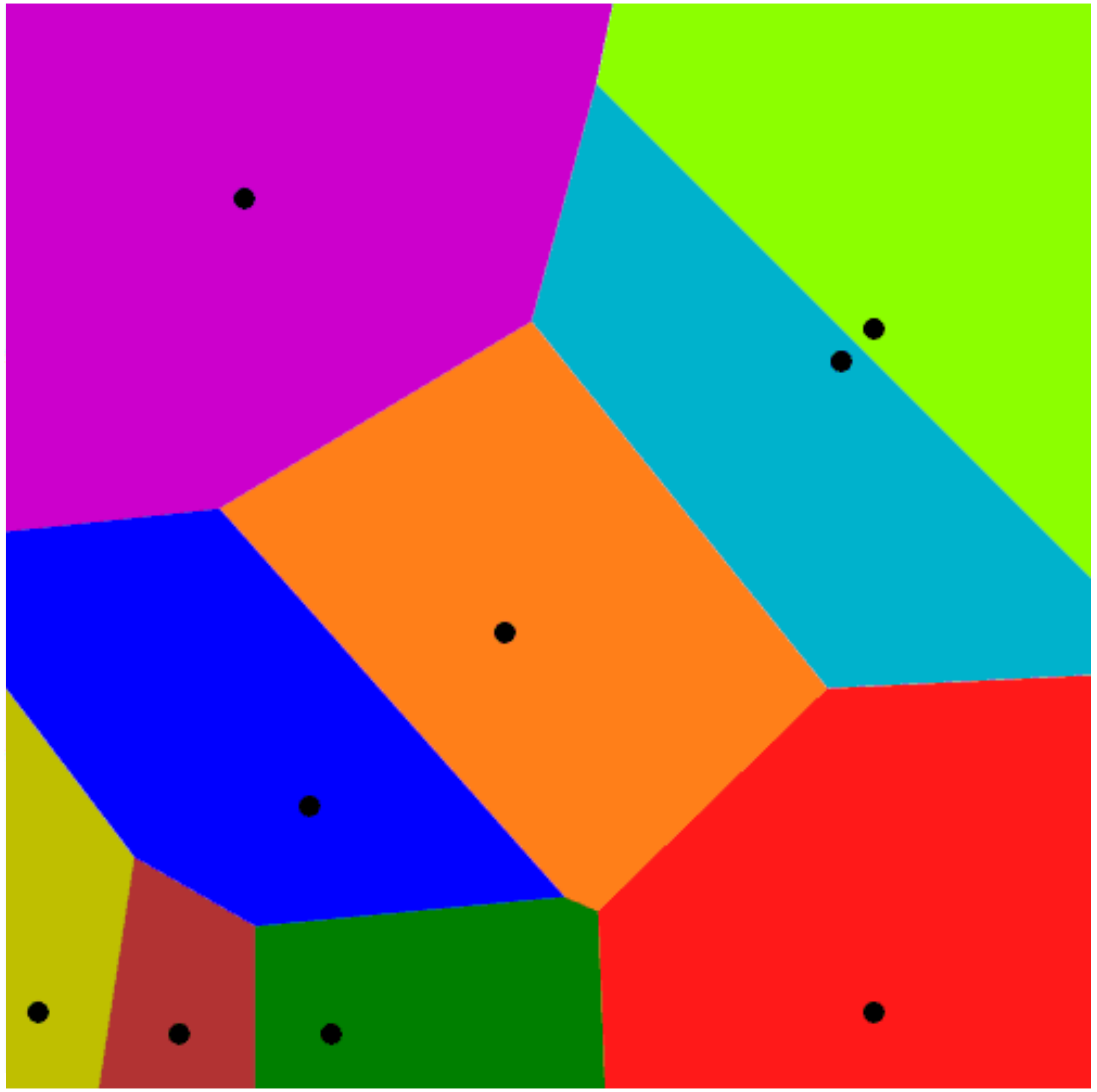
Voronoi Diagram

- Given *N* input points in a *d* dimensional space
- Find region boundaries such that each point on a boundary are equidistant to two or more input points
- Delaunay triangulation is a dual to the Voronoi diagram



https://en.wikipedia.org/wiki/Voronoi_diagram#/media/File:Voronoi_growth_euclidean.gif

2 Approache

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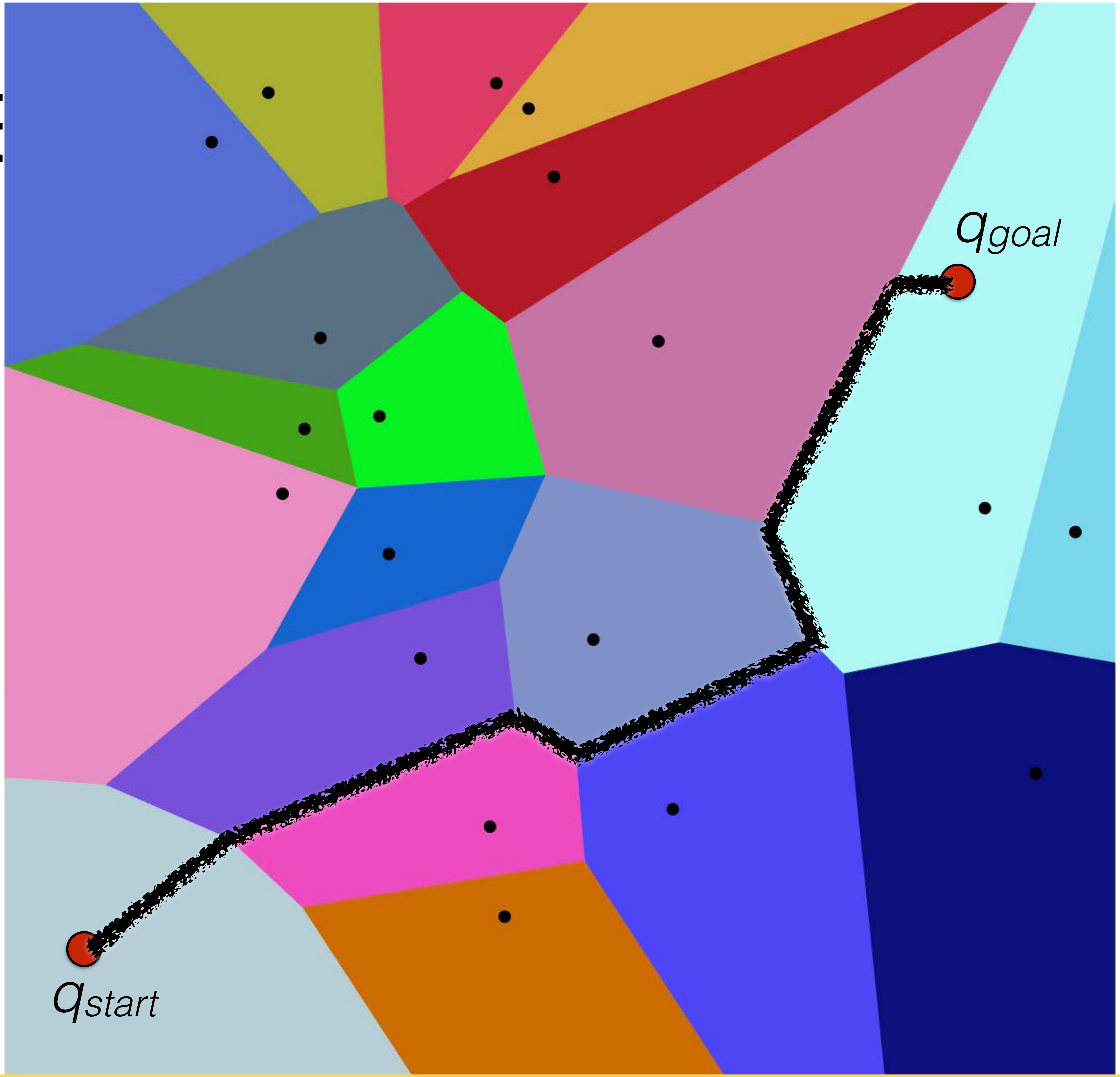

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2 Approaches to Roadmaps

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Probabilistic:

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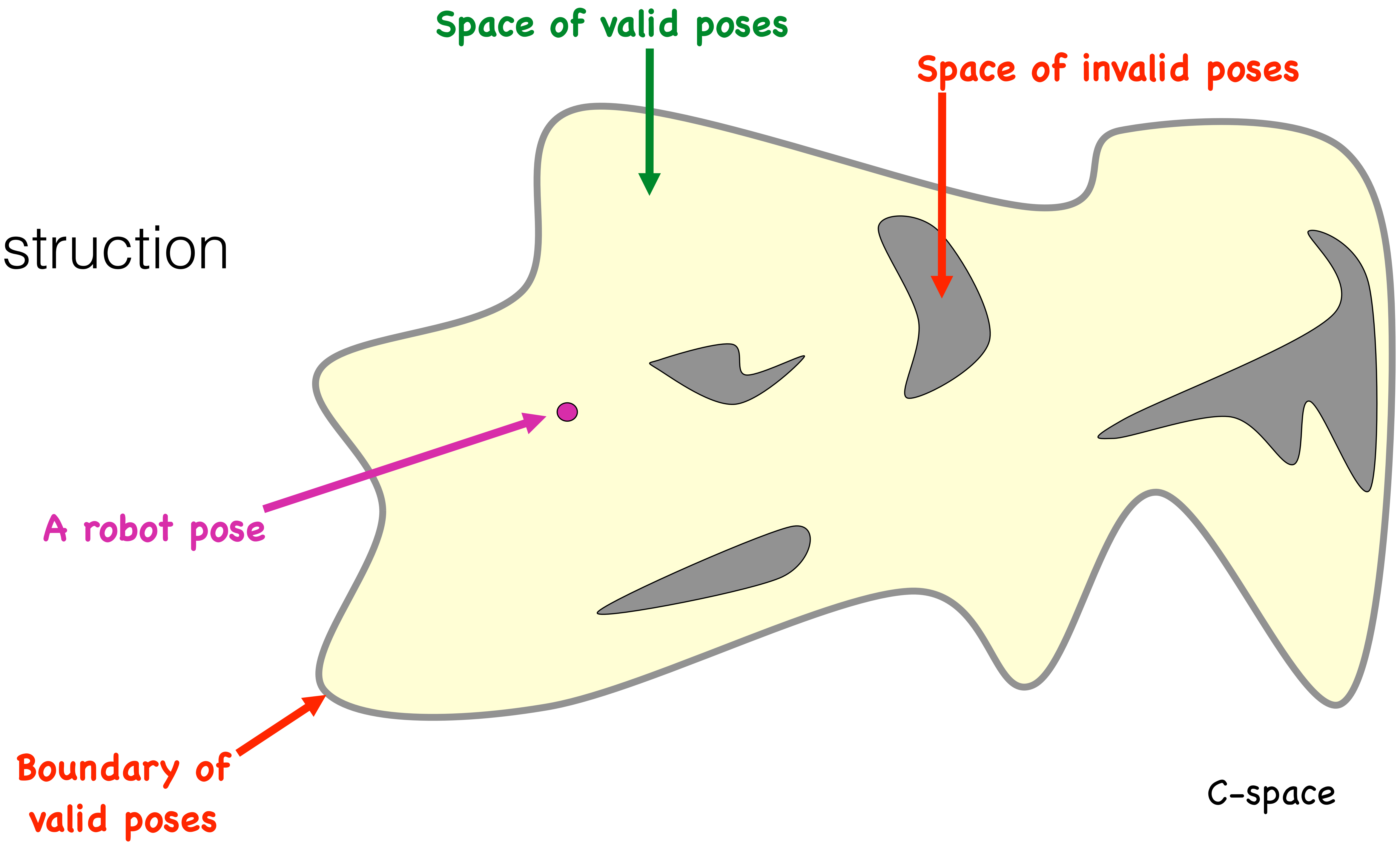
Probabilistic road maps

- Two phases
  - Roadmap construction
  - Path Query

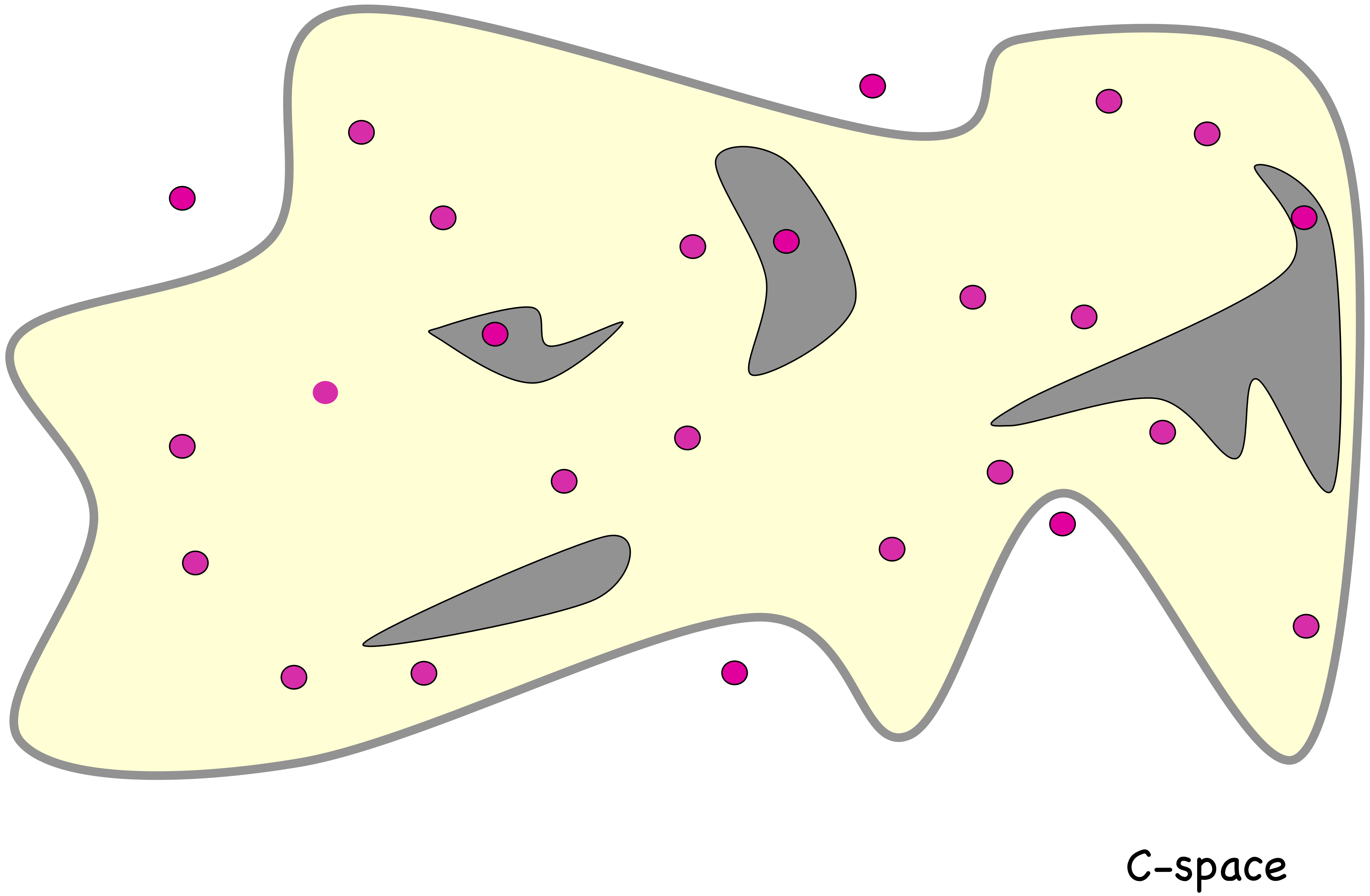


PRM: construction phase

1) **Select N sample poses at random**

2) Eliminate invalid poses

3) Connect neighboring poses



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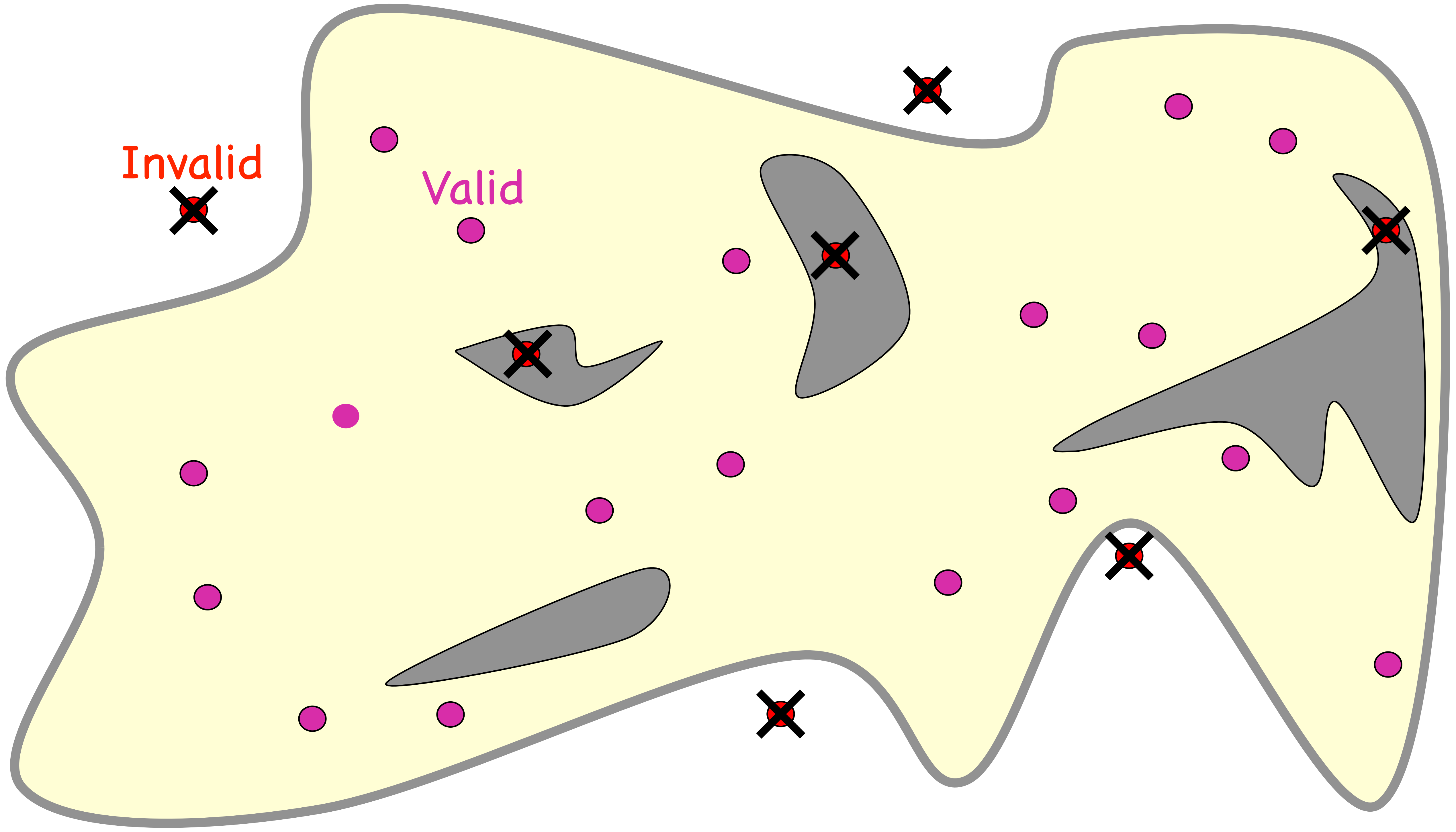

Collision detection will be covered later

C-space

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PRM: construction phase

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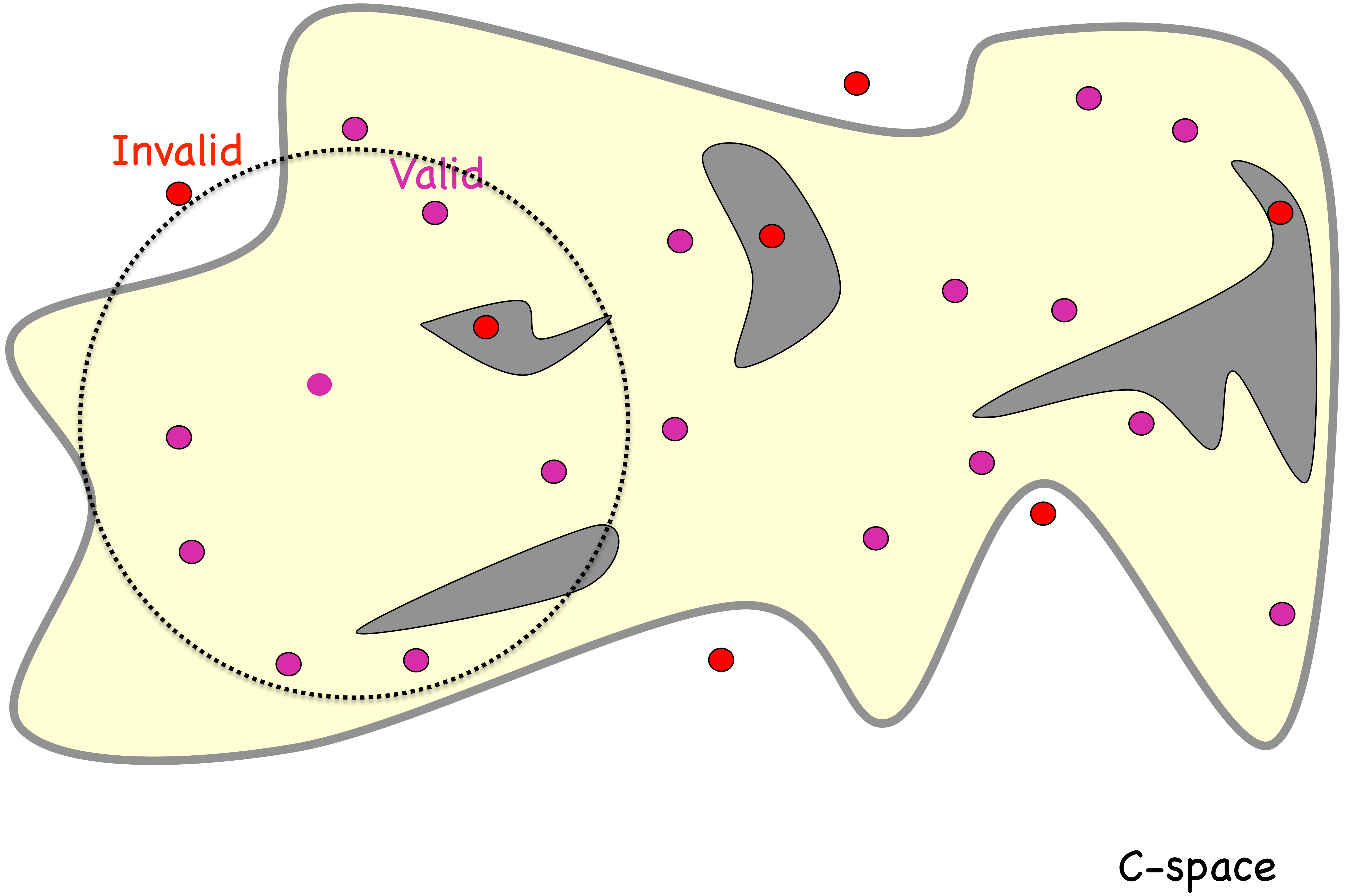
2) Eliminate invalid poses

3) **Connect neighboring poses**



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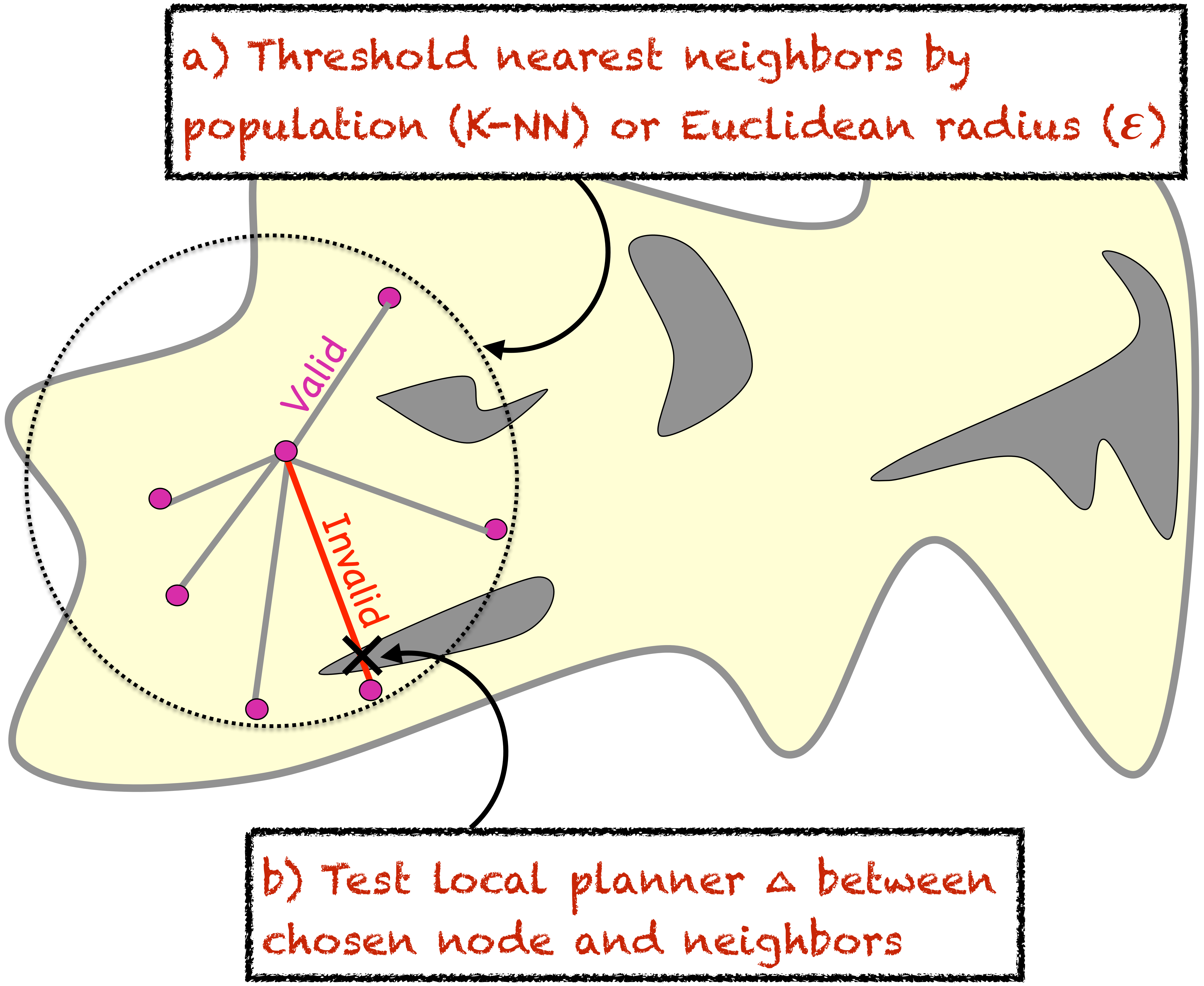


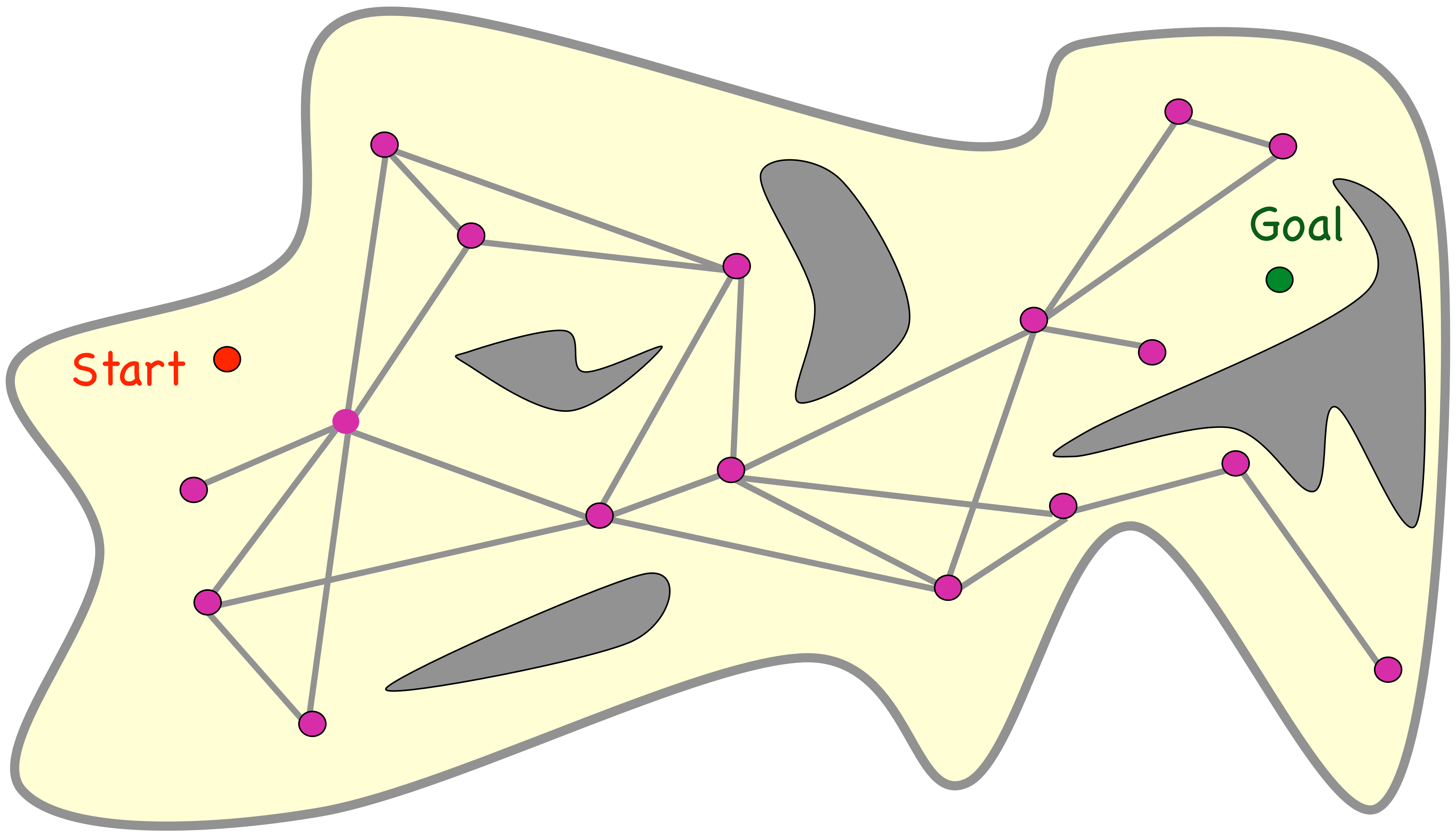
PRM: query phase

1) **Given constructed roadmap, start pose, and goal pose**

2) Attach goal and start to nearest roadmap entry nodes

3) Search for path between roadmap entry nodes

4) Return path with entry and departure edges



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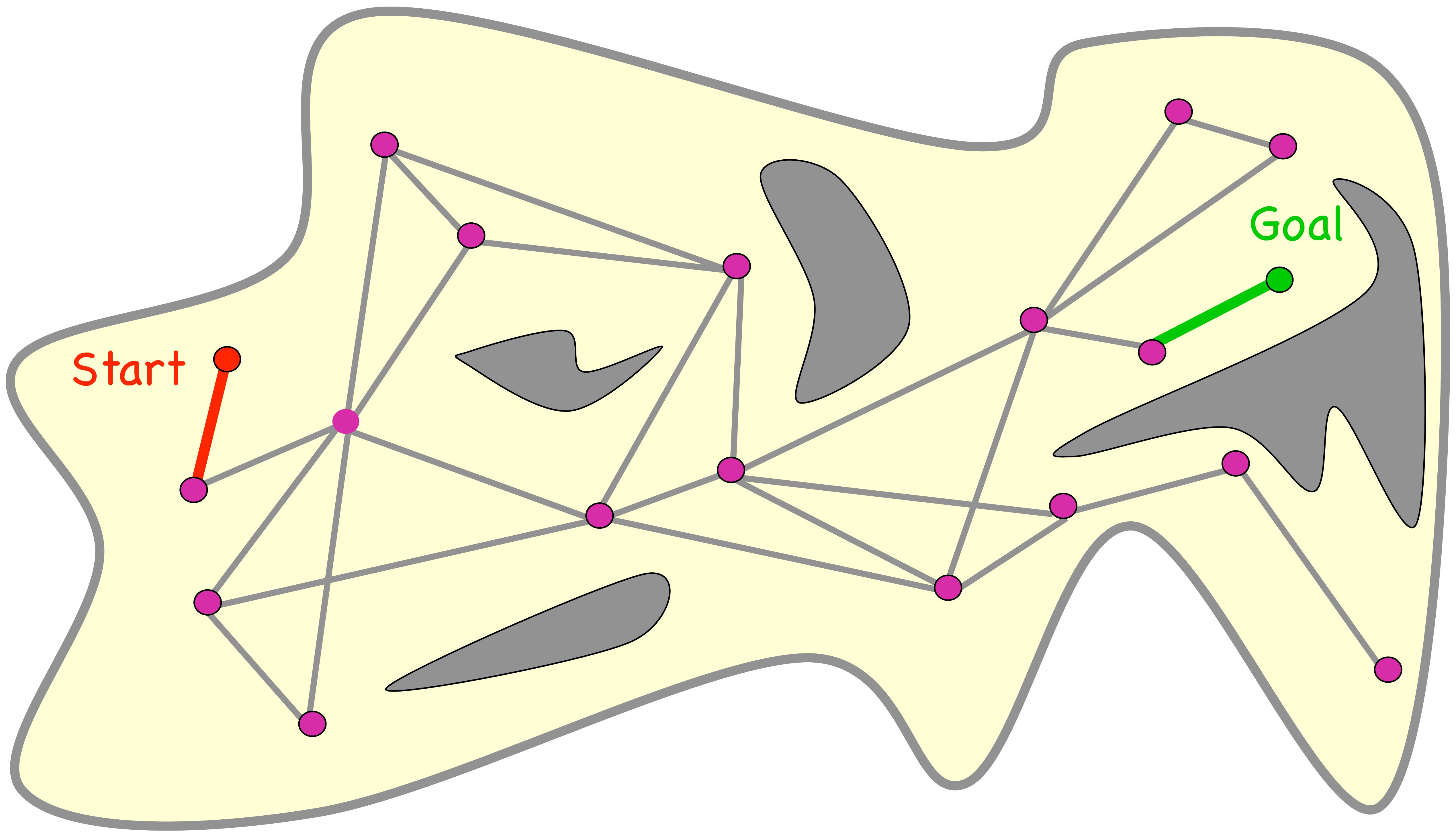

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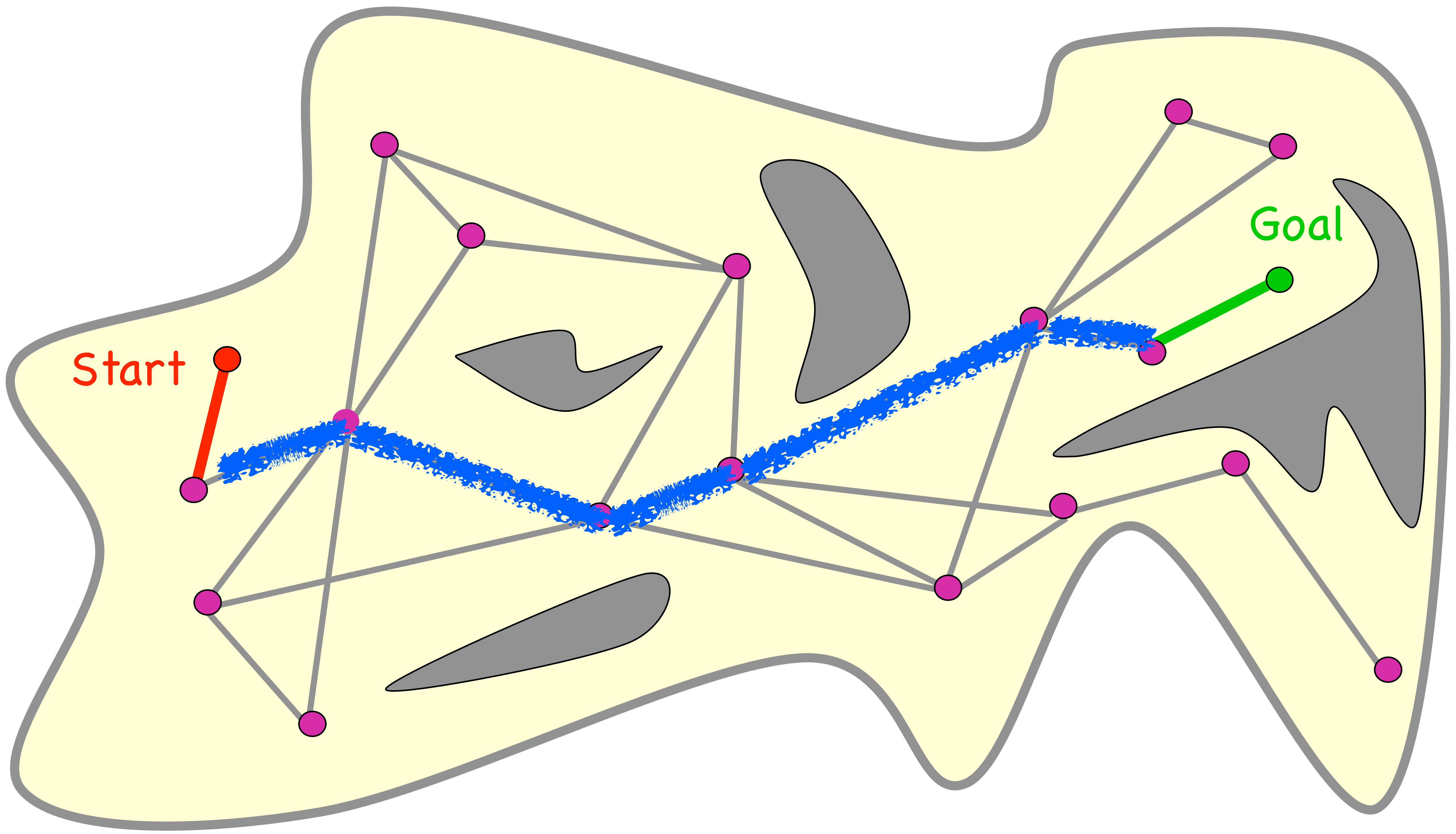

Remember: graph search algorithms
A*, Dijkstra, BFS, DFS

PRM: query phase

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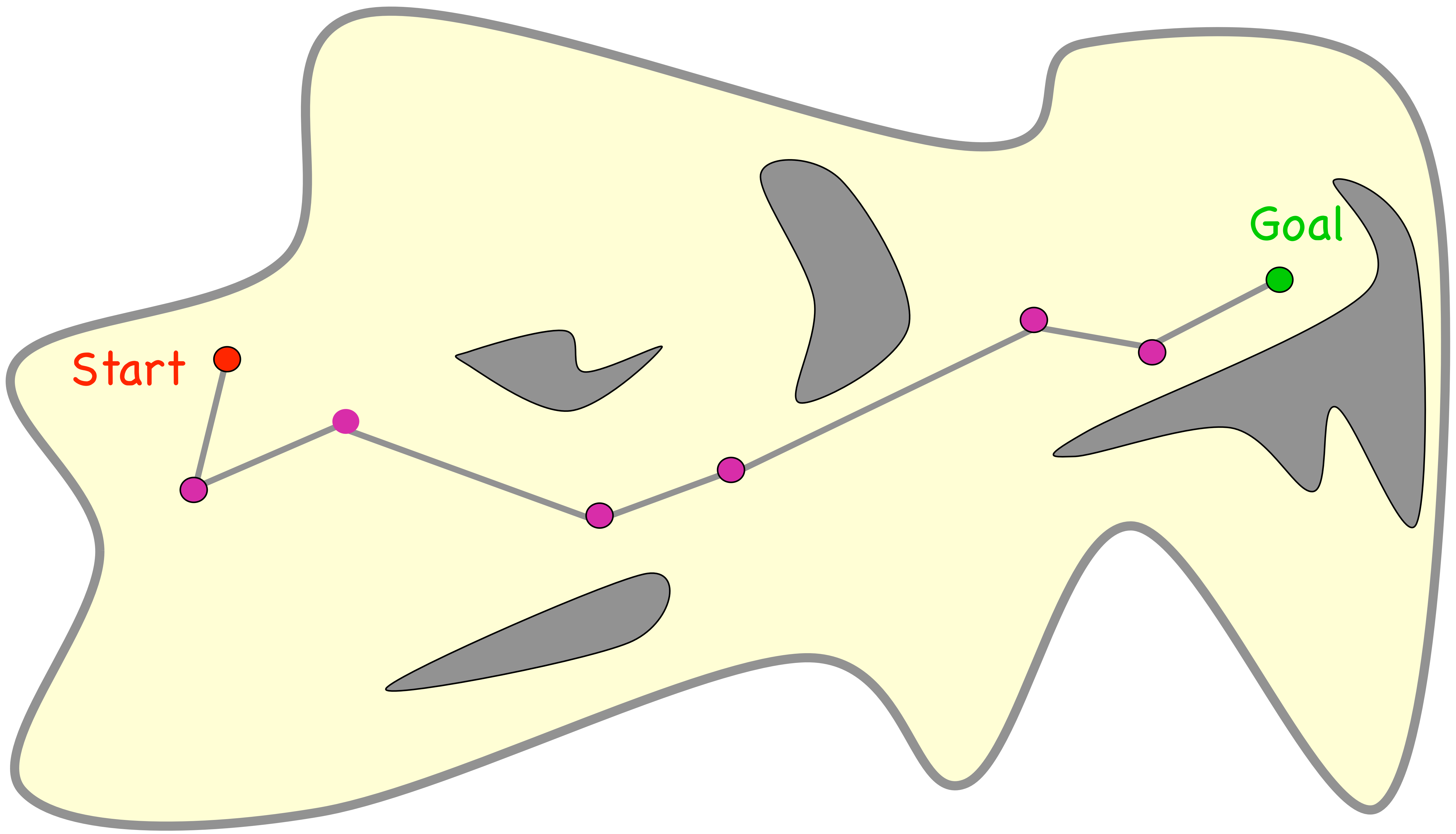

Multi-query planning: Considerations

i.e. if you will be querying the map multiple times (PRM by design allows this)

- Number of samples wrt. C-space dimensionality
- Balanced sampling over C-space
- Choice of distance (e.g., Euclidean)
- Choice of local planner (e.g., line subdivision)
- Selecting neighbors: (e.g., K-NN, kd-tree, cell hashing)

2 Approaches to Roadmaps

Deterministic:

complete algorithms

- Visibility Graph
  - trace lines connecting obstacle polygon vertices
- Voronoi Planning
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Probabilistic:

C-space sampling

- Probabilistic Roadmap (PRM)
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Single Query Planning

- Given specific start and goal configurations
- Grow trees from start and goal towards each other
- Path is found once trees connect
- Focus sampling in unexplored areas of C-space and moving towards start/goal
- Common algorithms:
    - ESTs (expansive space trees)
    - **RRTs (rapidly exploring random trees)**

RRT Algorithm

RRT Algorithm

Extend graph towards a random configuration and repeat

```
BUILD_RRT(q_init)
    T.init(q_init);
    for k = 1 to K do
        q_rand ← RANDOM_CONFIG();
        EXTEND(T, q_rand);
    Return T
```

[Kuffner, LaValle 2000]

RRT Algorithm

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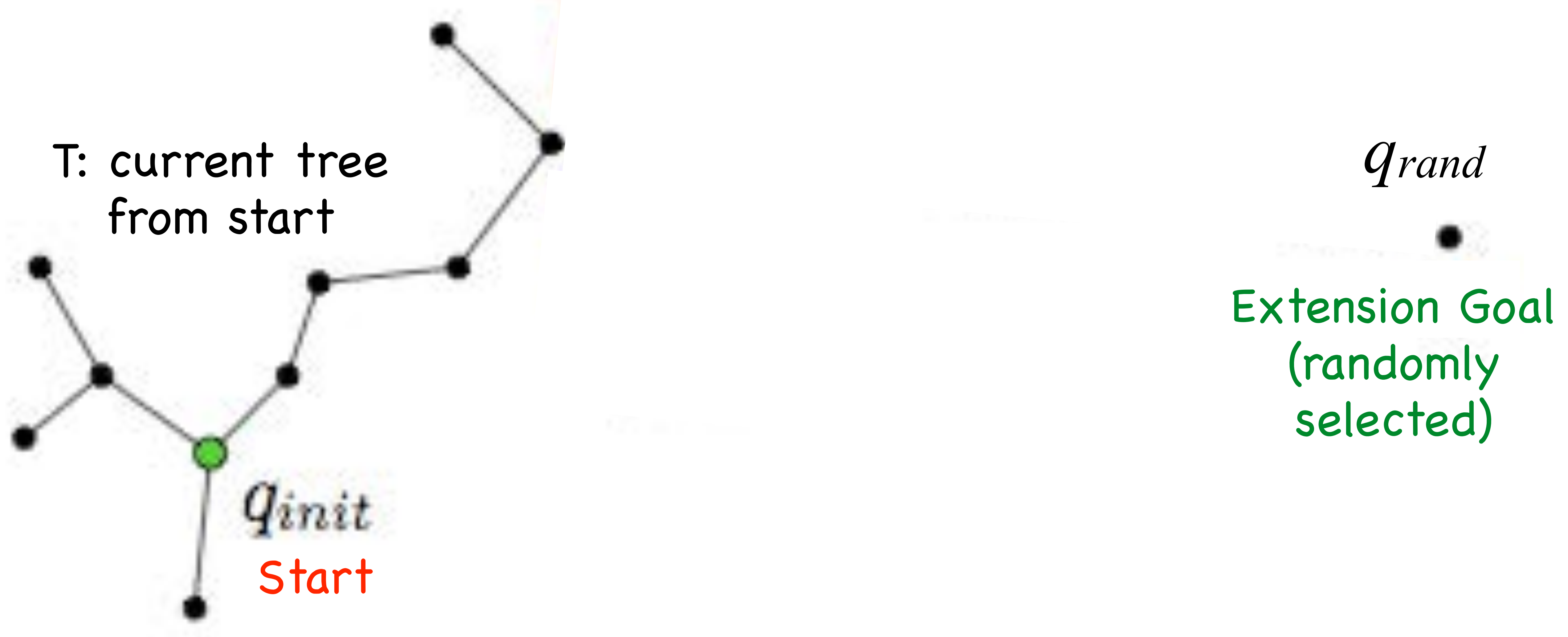

Figure 3: The EXTEND operation.

Slide borrowed from Michigan Robotics autorob.org

[Kuffner, LaValle 2000]

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  Return T
```

```
EXTEND(T, q)
  q_near ← NEAREST_NEIGHBOR(q, T);
  if NEW_CONFIG(q, q_near, q_new) then
      T.add_vertex(q_new);
      T.add_edge(q_near, q_new);
      if q_new = q then
          Return Reached;
      else
          Return Advanced;
  Return Trapped;
```

Extend graph towards a random configuration

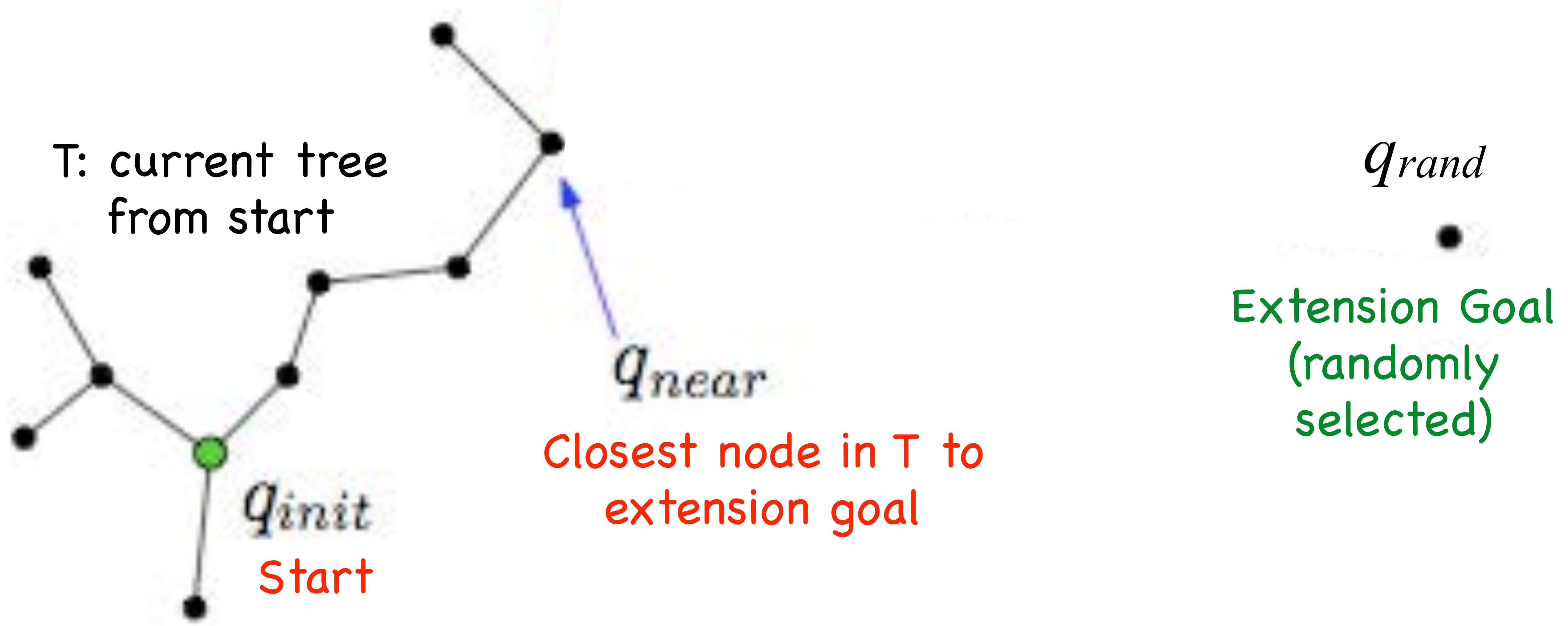


Figure 3: The EXTEND operation.

Slide borrowed from Michigan Robotics autorob.org

[Kuffner, LaValle 2000]

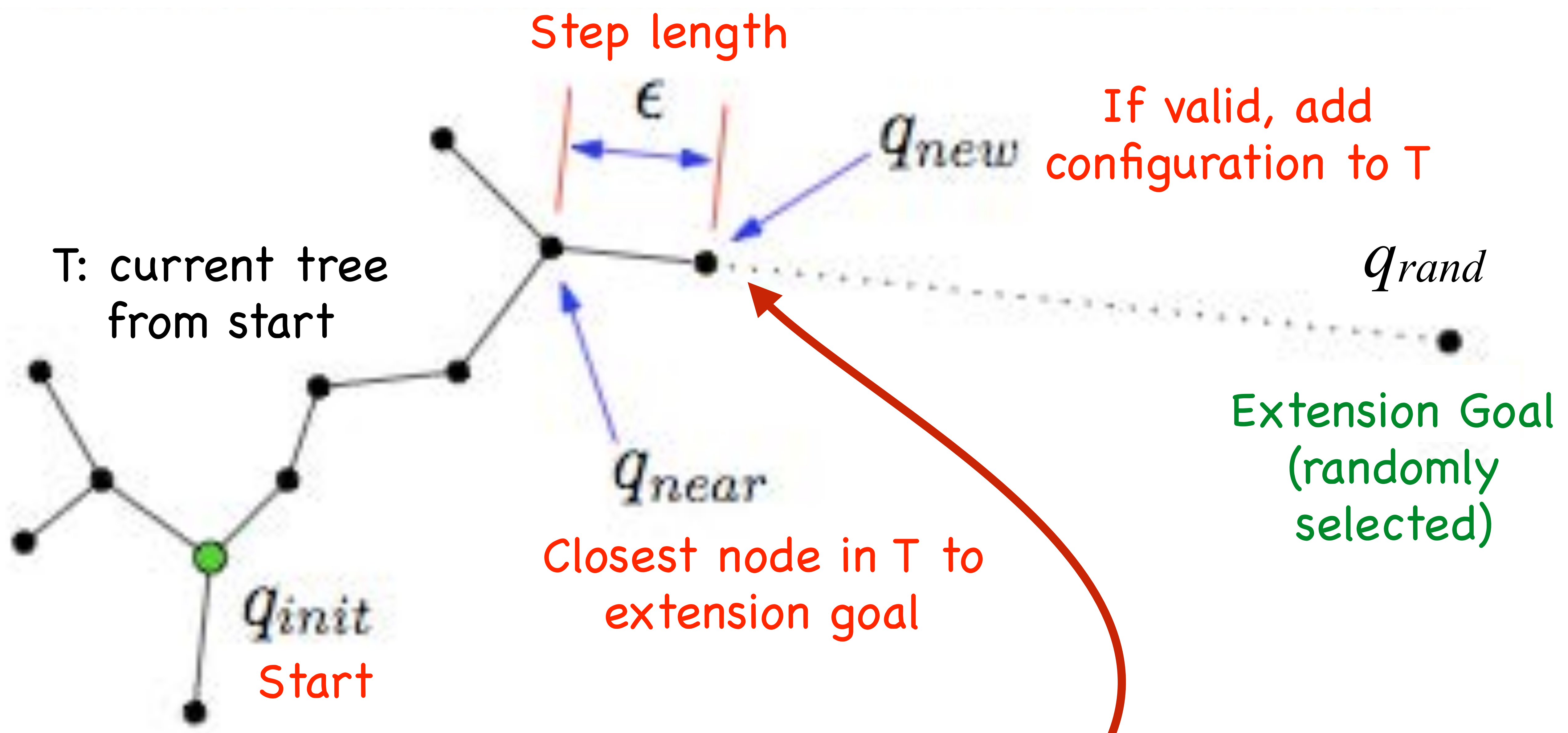
RRT Algorithm

Extend graph towards a random configuration and repeat

```
BUILD_RRT(q_init)
  T.init(q_init);
  for k = 1 to K do
      q_rand ← RANDOM_CONFIG();
      EXTEND(T, q_rand);
  Return T
```

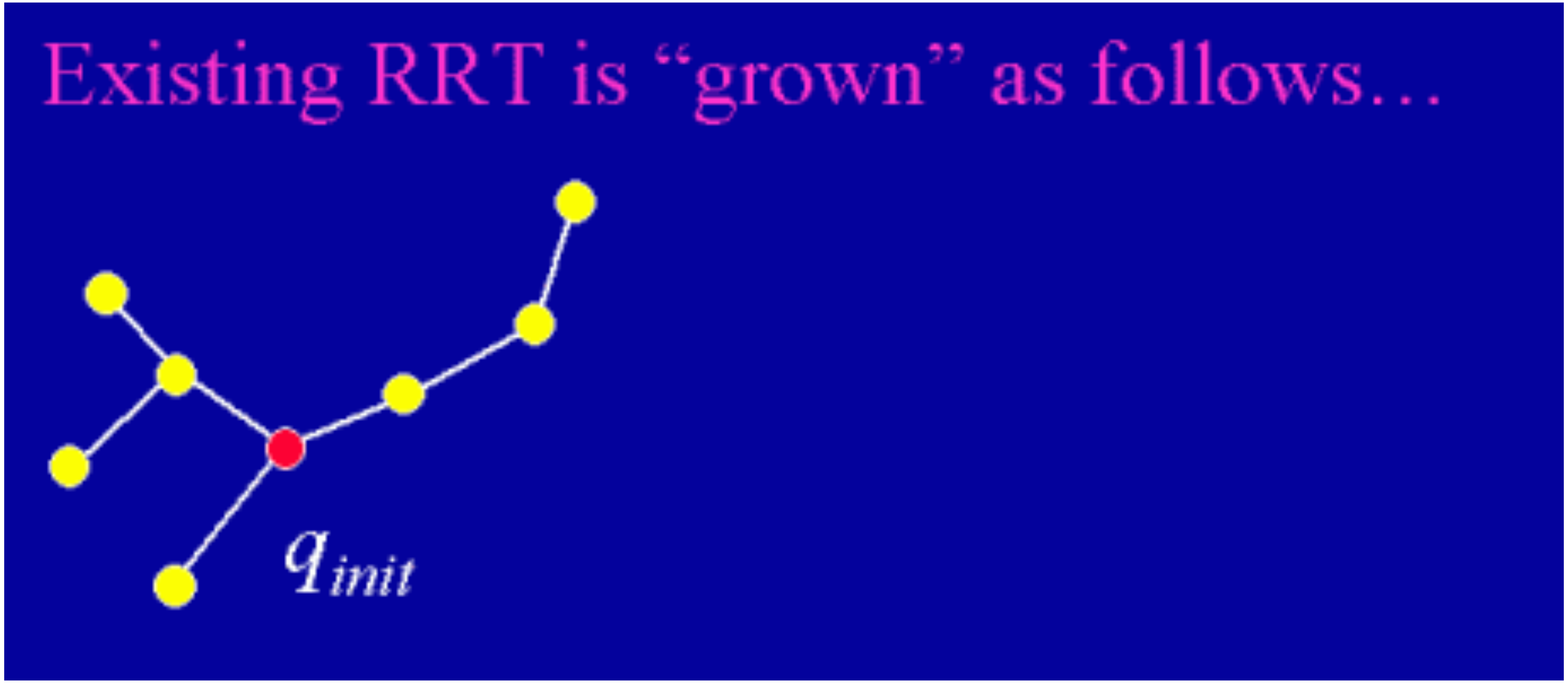
```
EXTEND(T, q)
  q_near ← NEAREST_NEIGHBOR(q, T);
  if NEW_CONFIG(q, q_near, q_new) then
      T.add_vertex(q_new);
      T.add_edge(q_near, q_new);
      if q_new = q then
          Return Reached;
      else
          Return Advanced;
  Return Trapped;
```

Extend graph towards a random configuration



Generate and test new configuration along vector in C-space from q_{near} to q_{rand}

RRT Extend animation

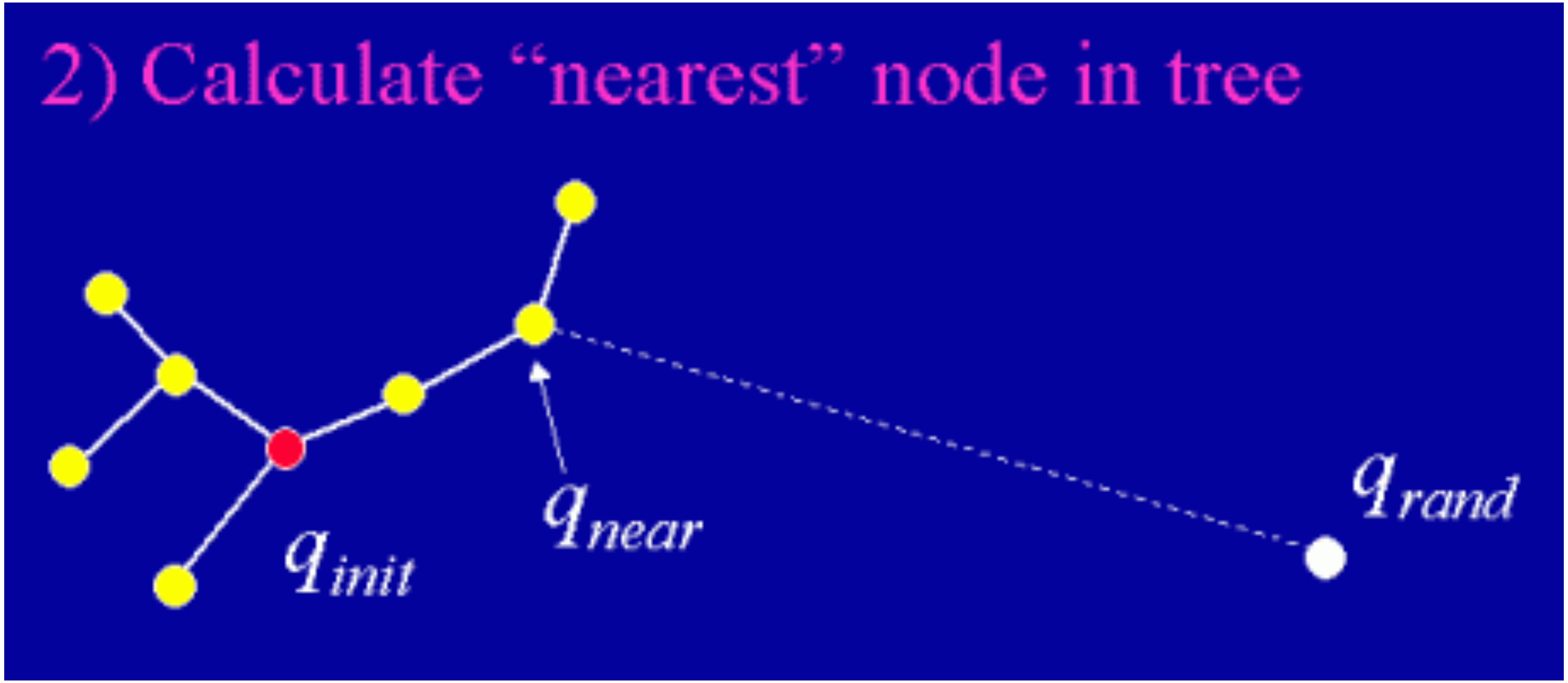


http://www.kuffner.org/james/plan/algorithm.php

RRT Extend animation

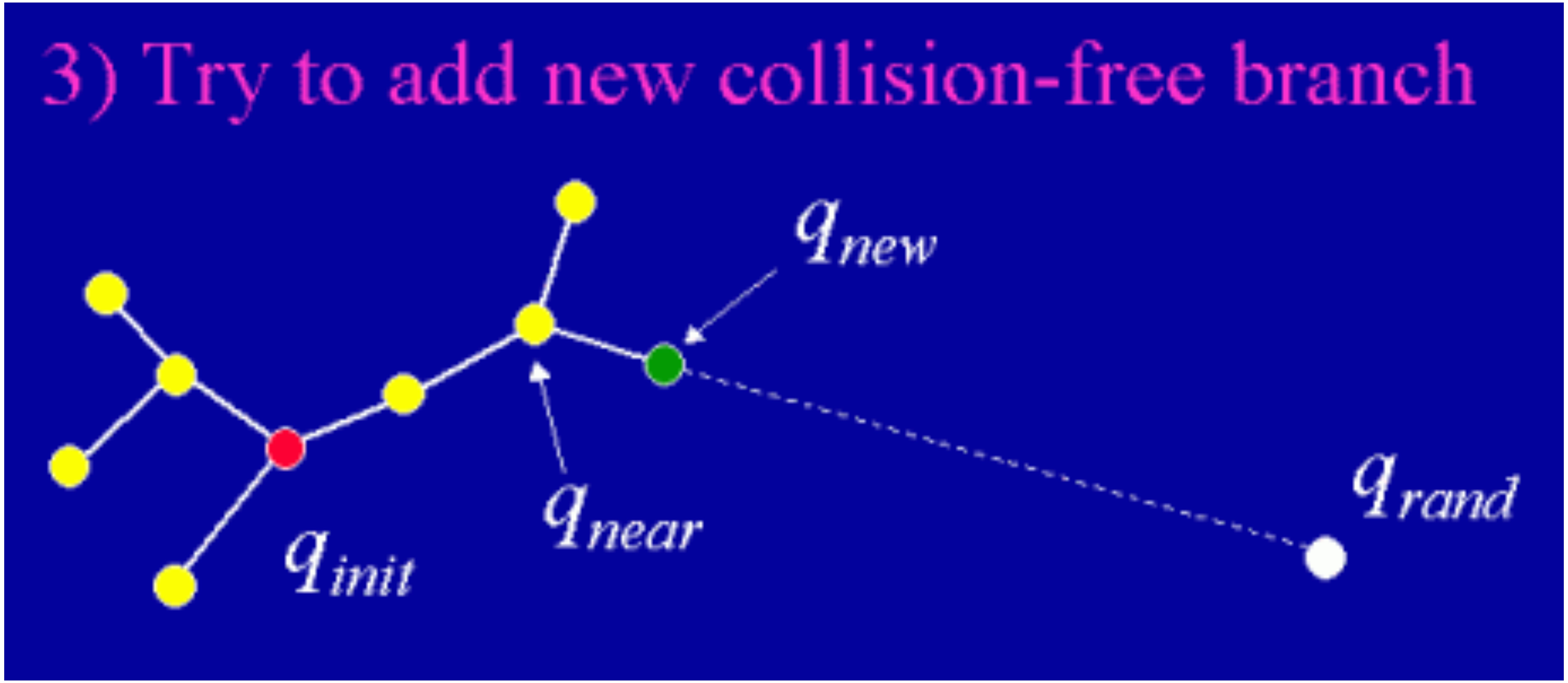
http://www.kuffner.org/james/plan/algorithm.php

RRT Extend animation



http://www.kuffner.org/james/plan/algorithm.php

RRT Extend animation



http://www.kuffner.org/james/plan/algorithm.php

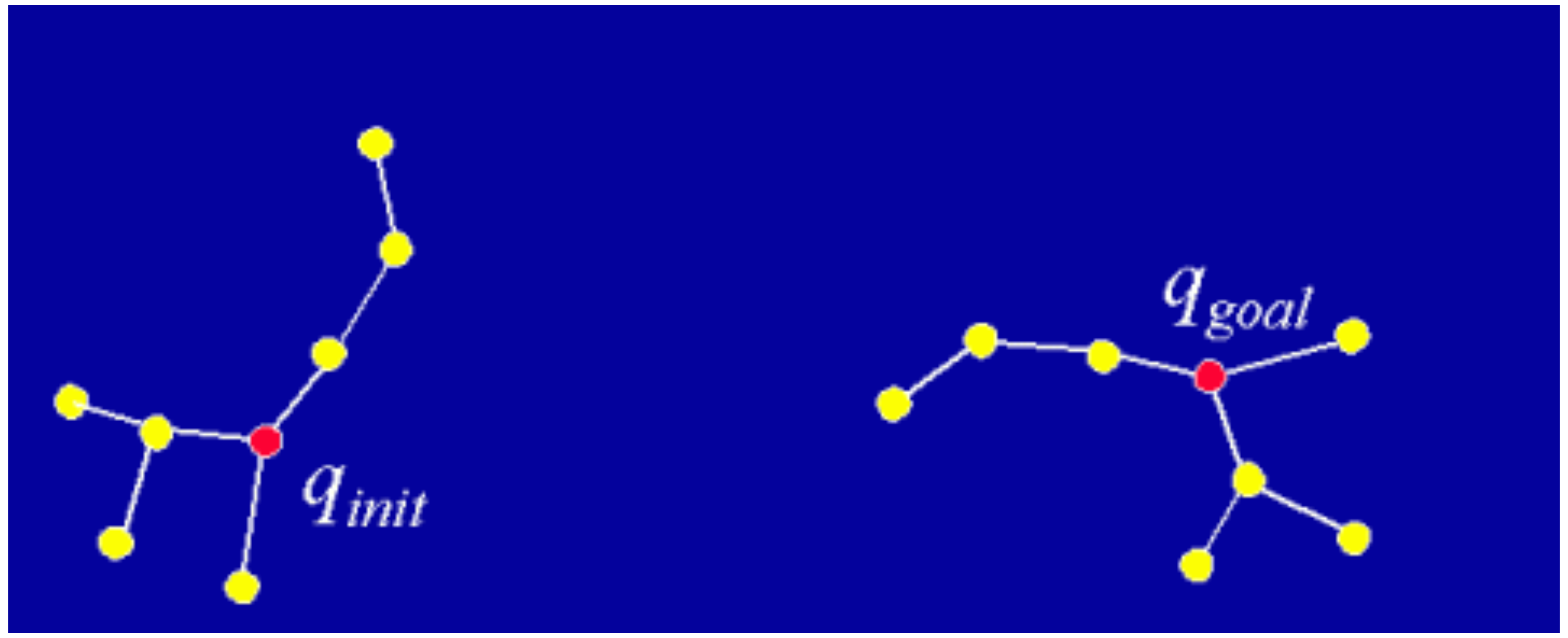
Demo

RRT Connect

[Kuffner, LaValle 2000]

0) Use 2 trees (A and B) rooted at start and goal configurations

```
RRT_CONNECT_PLANNER(q_init, q_goal)
T_a.init(q_init); T_b.init(q_goal);
for k = 1 to K do
    q_rand ← RANDOM_CONFIG();
    if not (EXTEND(T_a, q_rand) = Trapped) then
        if (CONNECT(T_b, q_new) = Reached) then
            Return PATH(T_a, T_b);
    SWAP(T_a, T_b);
Return Failure
```



RRT Connect

[Kuffner, LaValle 2000]

0) Use 2 trees (A and B) rooted at start and goal configurations

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RRT_CONNECT_PLANNER(q_init, q_goal)
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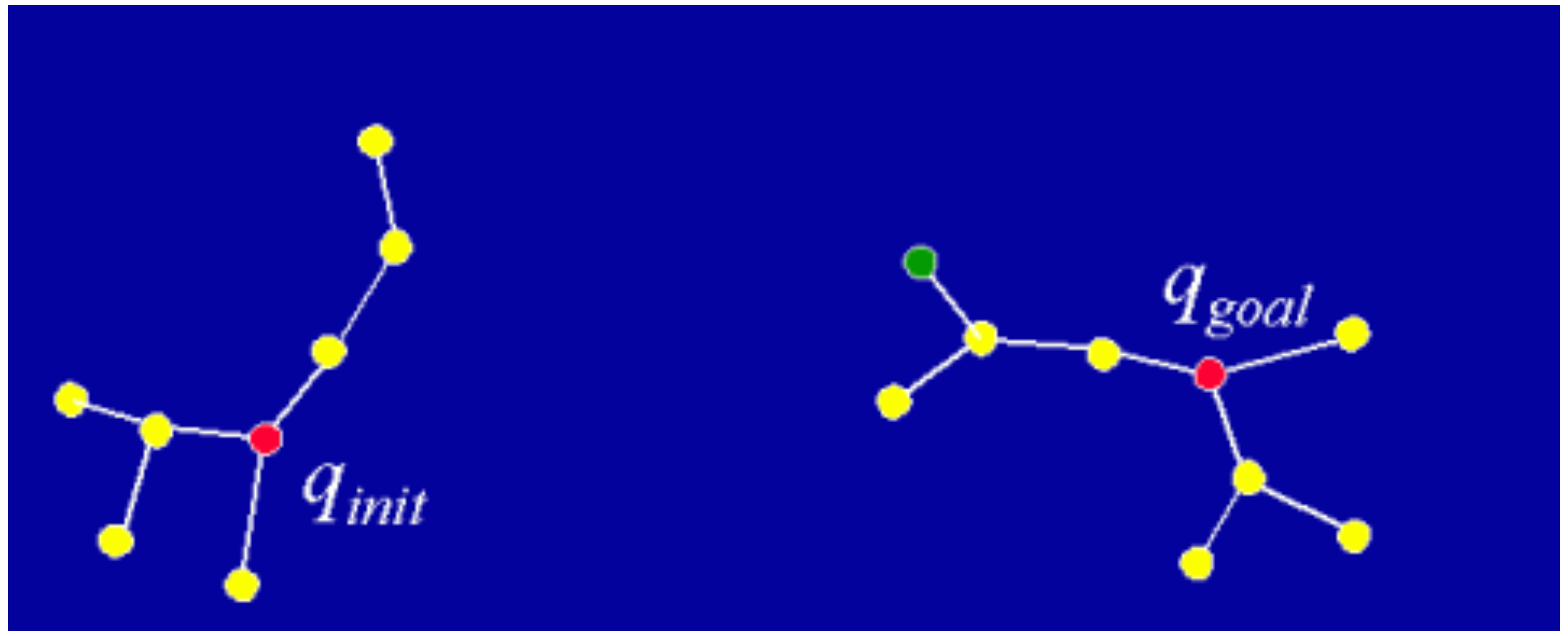
            if (CONNECT(T_b, q_new) = Reached) then
                Return PATH(T_a, T_b);
        SWAP(T_a, T_b);
    Return Failure
```

```
EXTEND(T, q)
    q_near ← NEAREST_NEIGHBOR(q, T);
    if NEW_CONFIG(q, q_near, q_new) then
        T.add_vertex(q_new);
        T.add_edge(q_near, q_new);
        if q_new = q then
            Return Reached;
        else
            Return Advanced;
    Return Trapped;
```

1) Extend tree A towards a random configuration



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RRT Connect

[Kuffner, LaValle 2000]

0) Use 2 trees (A and B) rooted at start and goal configurations

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RRT_CONNECT_PLANNER(q_init, q_goal)
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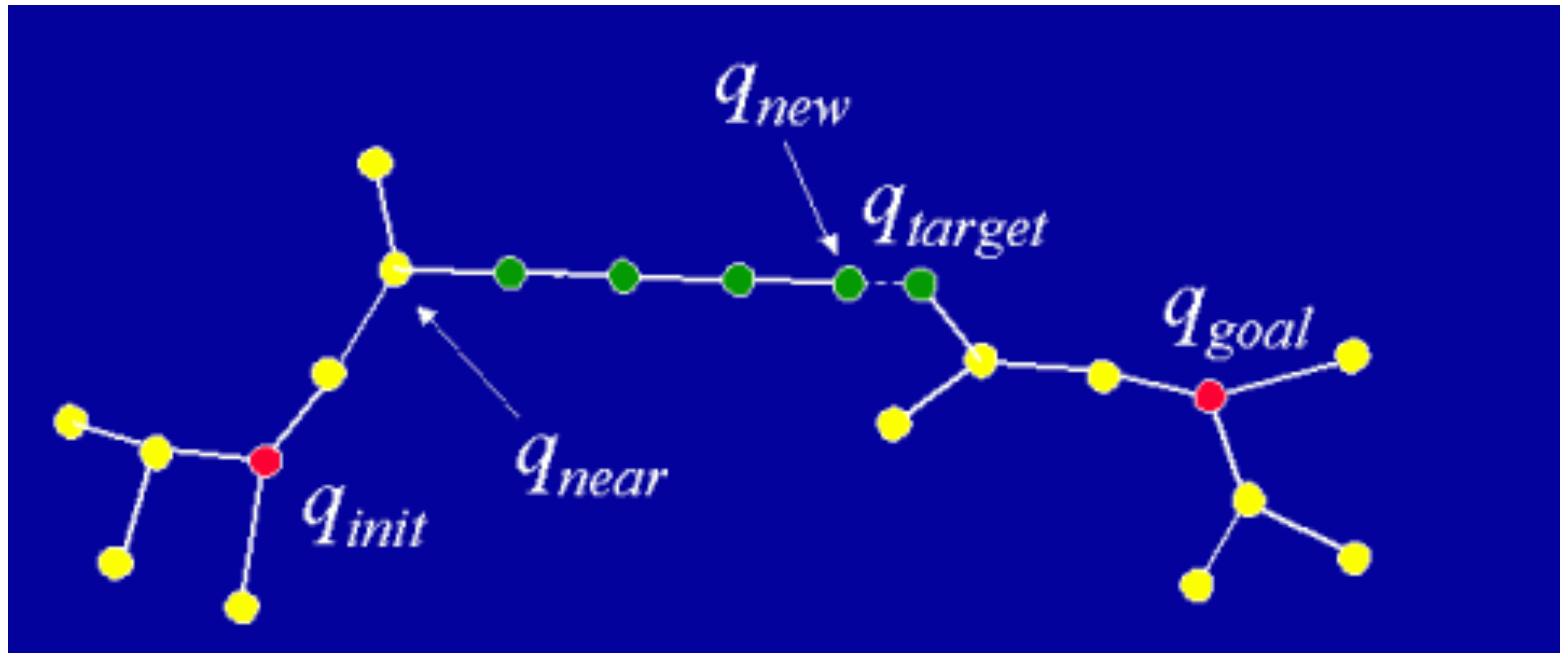
          if (CONNECT(T_b, q_new) = Reached) then
              Return PATH(T_a, T_b);
      SWAP(T_a, T_b);
  Return Failure
```

```
EXTEND(T, q)
  q_near ← NEAREST_NEIGHBOR(q, T);
  if NEW_CONFIG(q, q_near, q_new) then
      T.add_vertex(q_new);
      T.add_edge(q_near, q_new);
      if q_new = q then
          Return Reached;
      else
          Return Advanced;
  Return Trapped;
```

1) Extend tree A towards a random configuration

```
CONNECT(T, q)
  repeat
      S ← EXTEND(T, q);
  until not (S = Advanced)
  Return S;
```

2) Try to connect tree B to tree A by extending repeatedly from its nearest neighbor

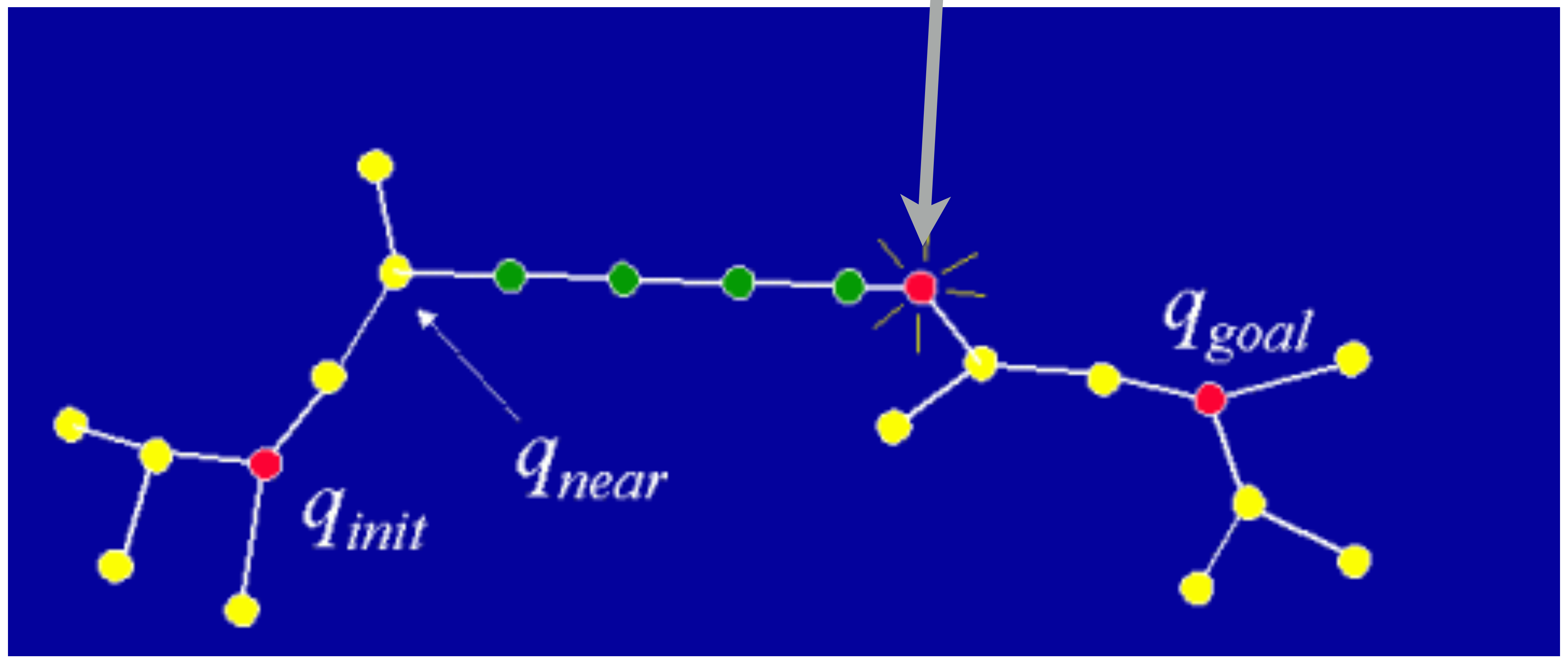


RRT Connect

0) Use 2 trees (A and B) rooted at start and goal configurations

RRT_CONNECT_PLANNER(q_{init}, q_{goal})
$\mathcal{T}_a$.init(q_{init}); $\mathcal{T}_b$.init(q_{goal});
for $k = 1$ **to** K **do**
  $q_{rand} \leftarrow$ RANDOM_CONFIG();
  if not (EXTEND($\mathcal{T}_a, q_{rand}$) $=$ *Trapped*) **then**
    if (CONNECT($\mathcal{T}_b, q_{new}$) $=$ *Reached*) **then**
      Return PATH($\mathcal{T}_a, \mathcal{T}_b$);
  SWAP($\mathcal{T}_a, \mathcal{T}_b$);
Return *Failure*

search succeeds if trees connect



EXTEND($\mathcal{T}, q$)
$q_{near} \leftarrow$ NEAREST_NEIGHBOR($q, \mathcal{T}$);
if NEW_CONFIG(q, q_{near}, q_{new}) **then**
  $\mathcal{T}$.add_vertex(q_{new});
  $\mathcal{T}$.add_edge(q_{near}, q_{new});
  if $q_{new} = q$ **then**
    Return *Reached*;
  else
    Return *Advanced*;
Return *Trapped*;

1) Extend tree A towards a random configuration

CONNECT($\mathcal{T}, q$)
repeat
  $S \leftarrow$ EXTEND($\mathcal{T}, q$);
until not ($S =$ *Advanced*)
Return S;

2) Try to connect tree B to tree A by extending repeatedly from its nearest neighbor

RRT Connect

[Kuffner, LaValle 2000]

0) Use 2 trees (A and B) rooted at start and goal configurations

```
RRT_CONNECT_PLANNER(q_init, q_goal)
  T_a.init(q_init); T_b.init(q_goal);
  for k = 1 to K do
      q_rand ← RANDOM_CONFIG();
      if not (EXTEND(T_a, q_rand) = Trapped) then
          if (CONNECT(T_b, q_new) = Reached) then
              Return PATH(T_a, T_b);
      SWAP(T_a, T_b);
  Return Failure
```

```
EXTEND(T, q)
  q_near ← NEAREST_NEIGHBOR(q, T);
  if NEW_CONFIG(q, q_near, q_new) then
      T.add_vertex(q_new);
      T.add_edge(q_near, q_new);
      if q_new = q then
          Return Reached;
      else
          Return Advanced;
  Return Trapped;
```

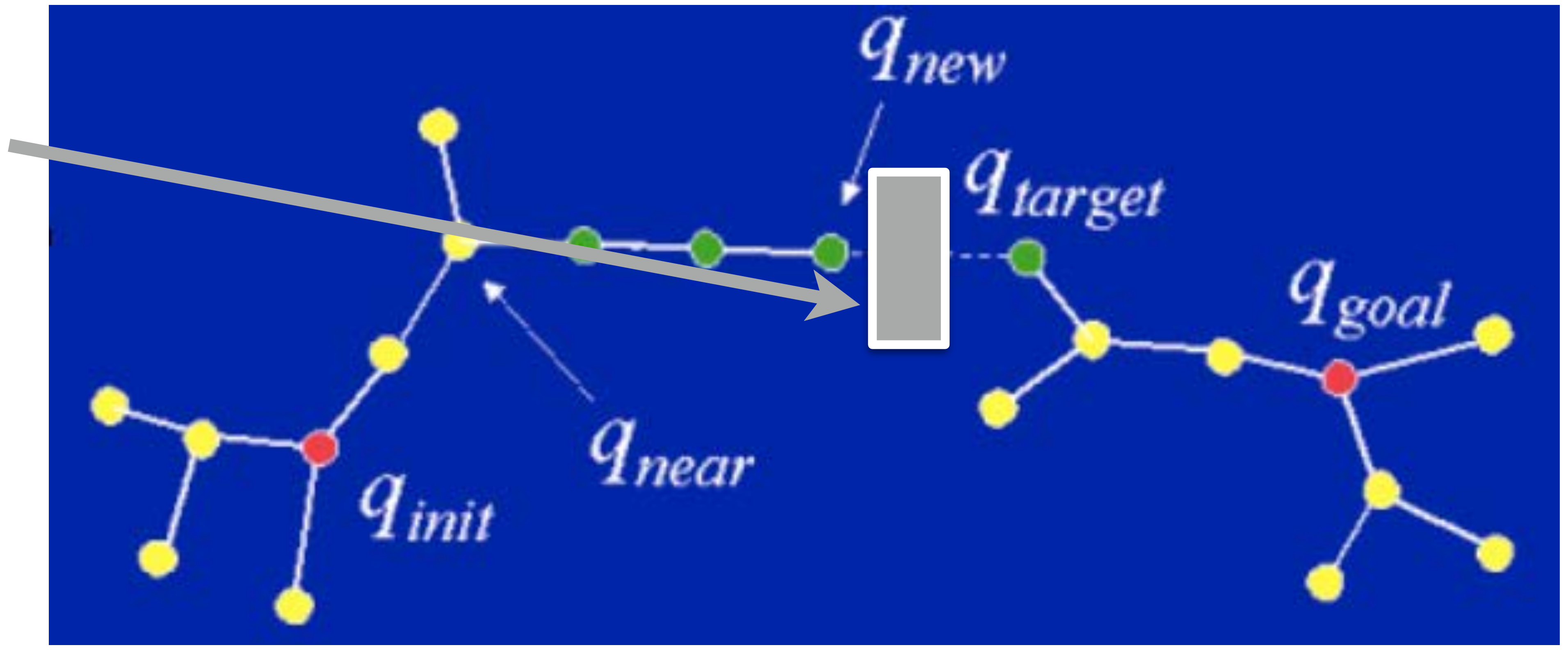
1) Extend tree A towards a random configuration

```
CONNECT(T, q)
  repeat
      S ← EXTEND(T, q);
  until not (S = Advanced)
  Return S;
```

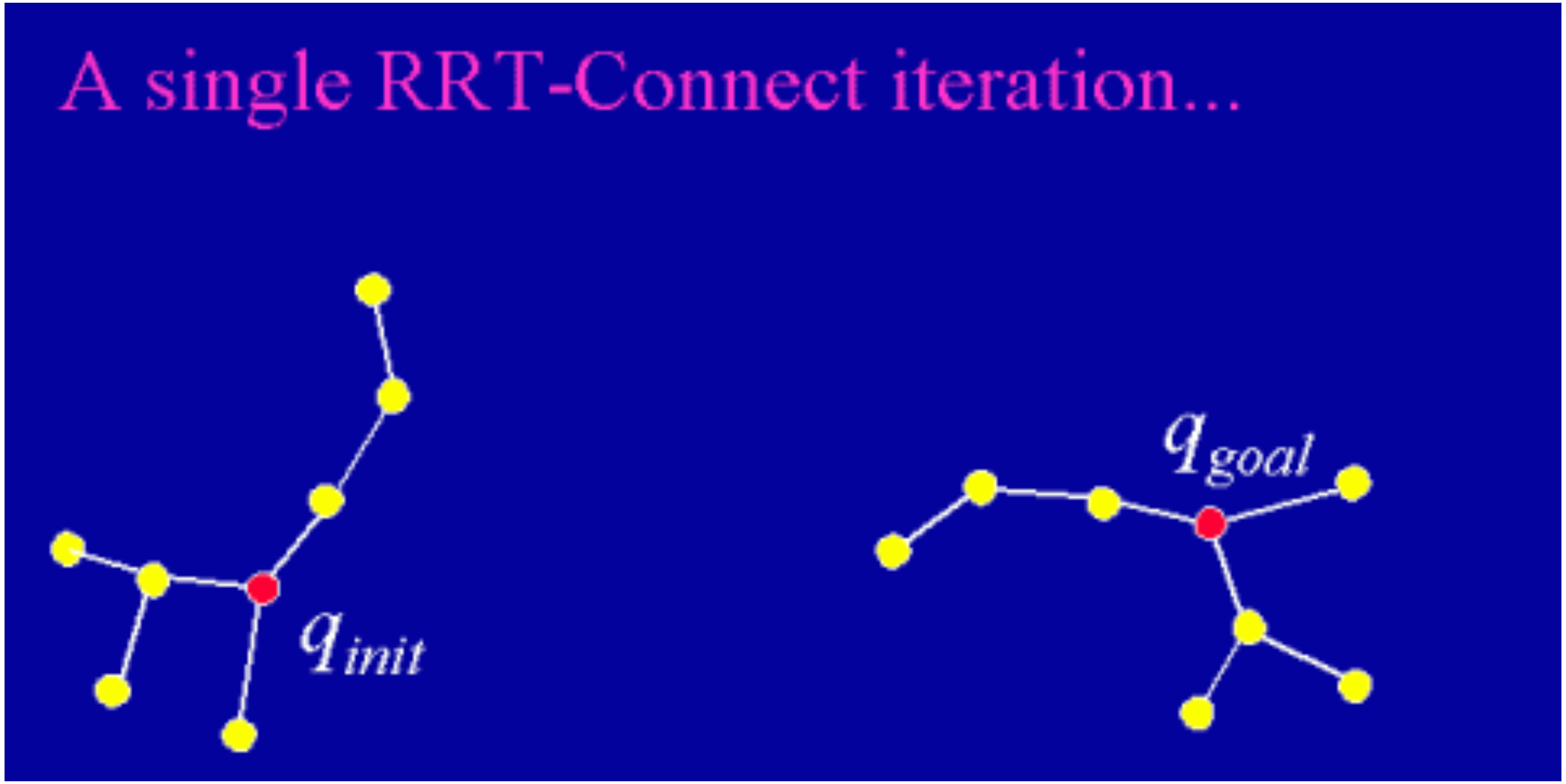
2) Try to connect tree B to tree A by extending repeatedly from its nearest neighbor

3) reverse roles for trees A and B and repeat

collision encountered

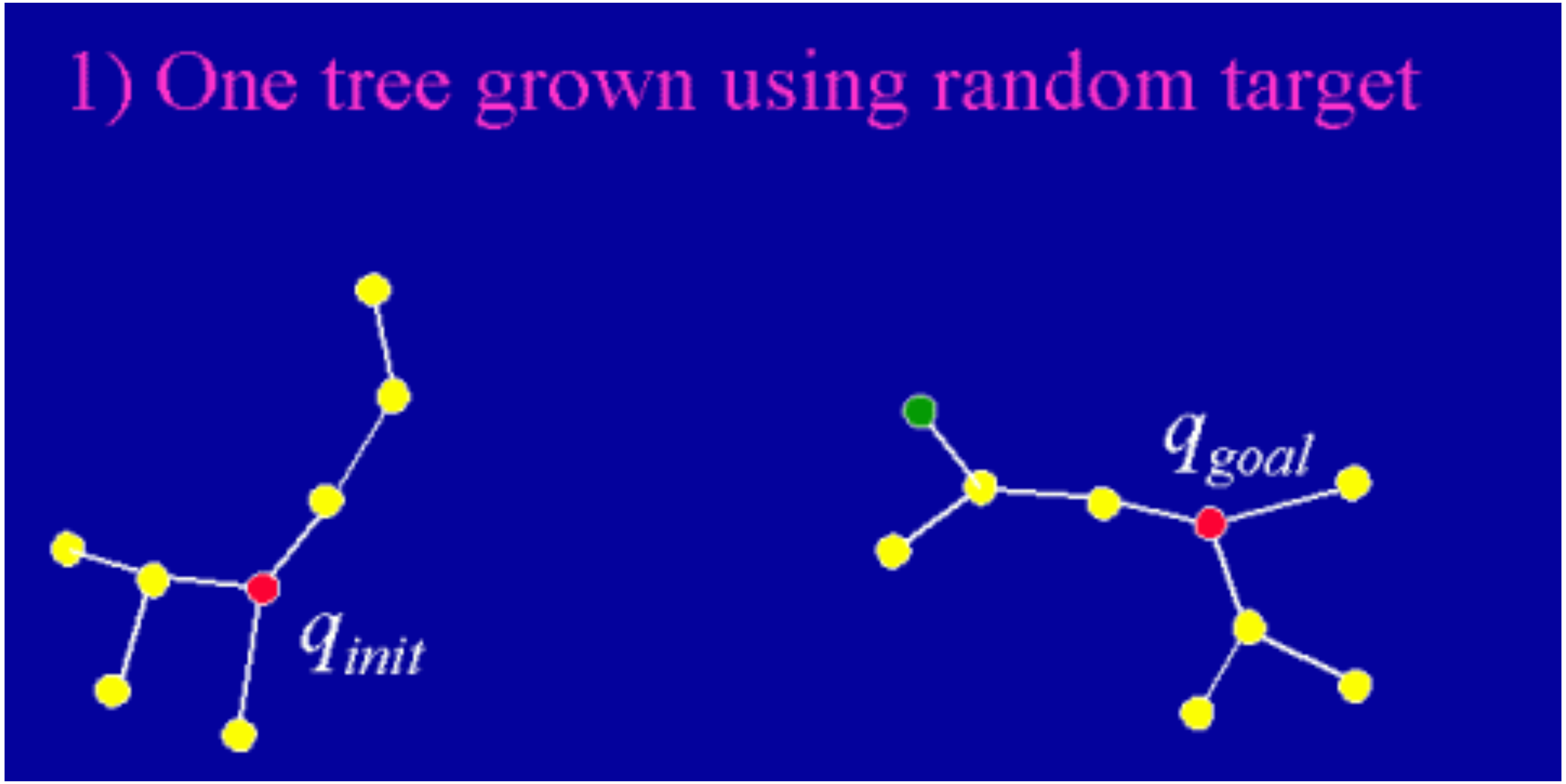


RRT-Connect animation



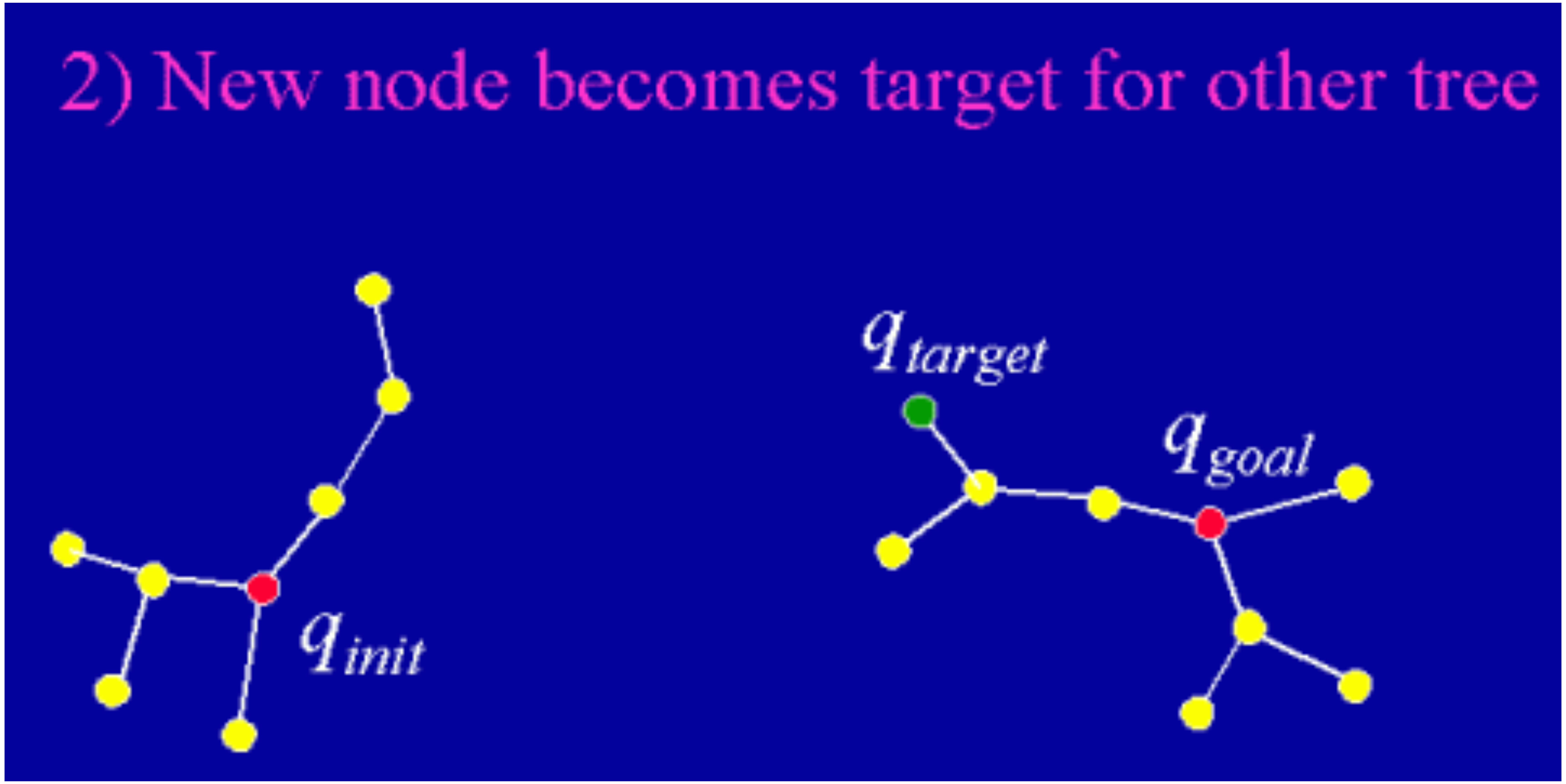
http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation



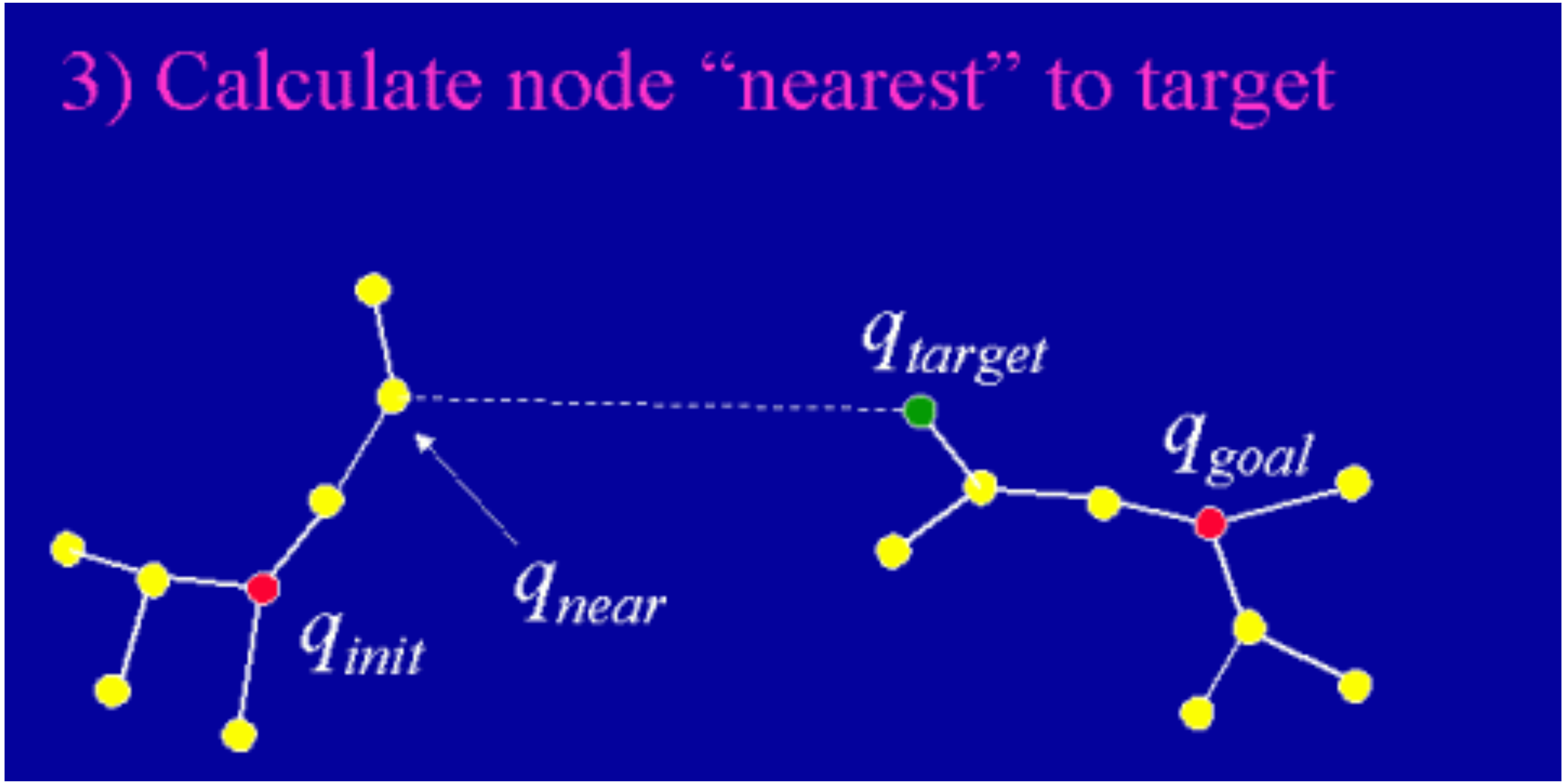
http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation



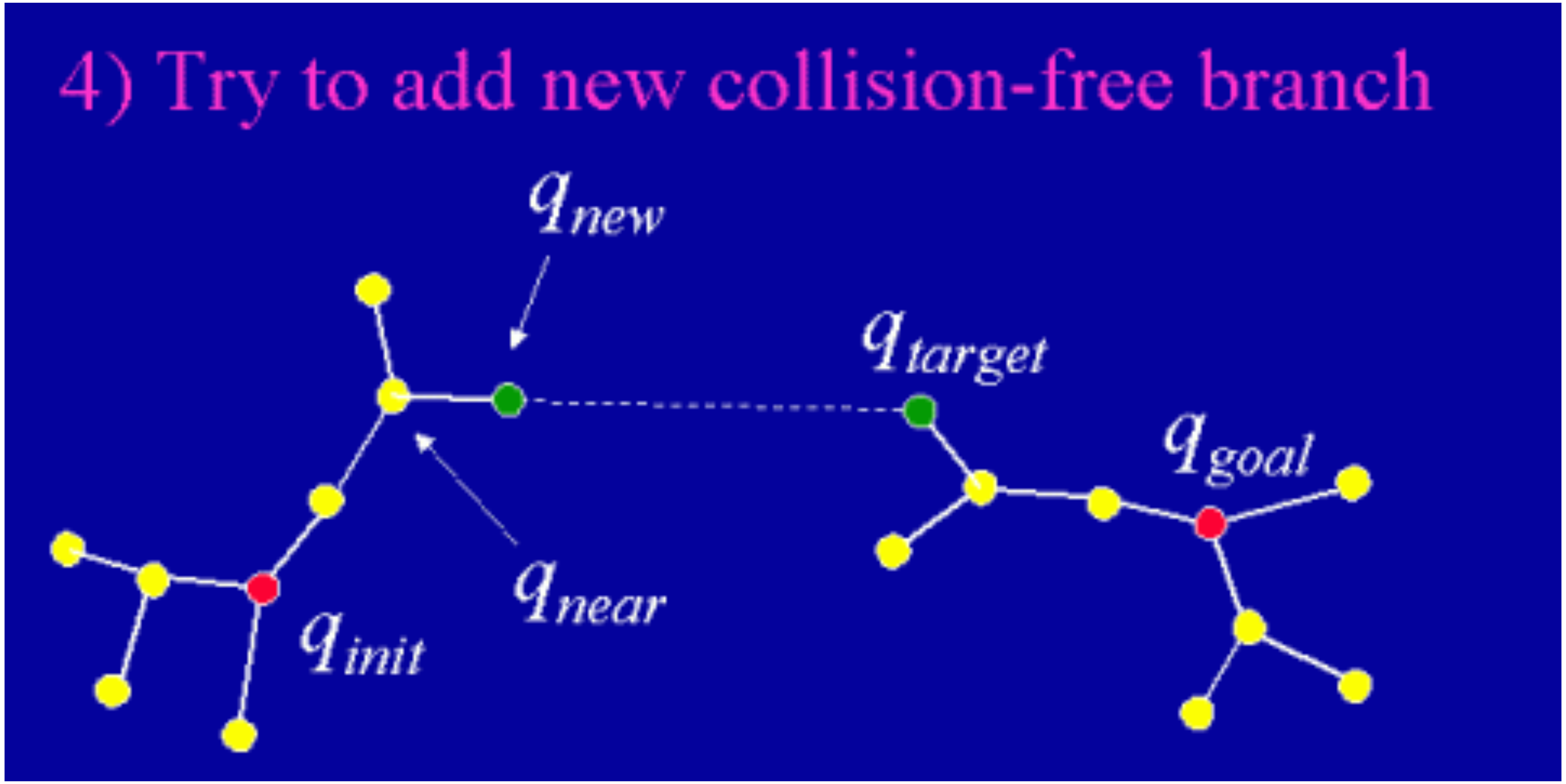
http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation



http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation



http://www.kuffner.org/james/plan/algorithm.php

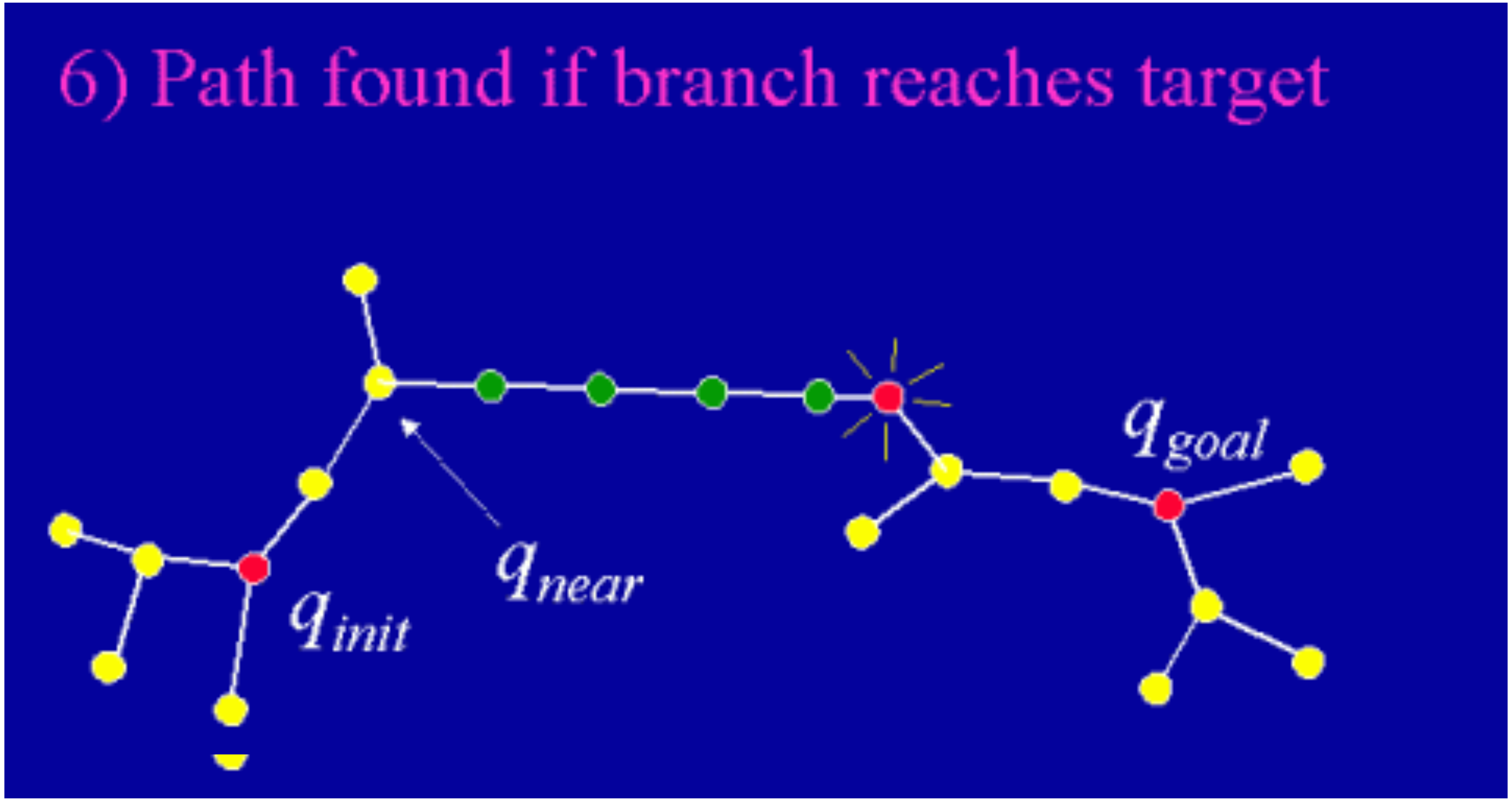
RRT-Connect animation

http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation

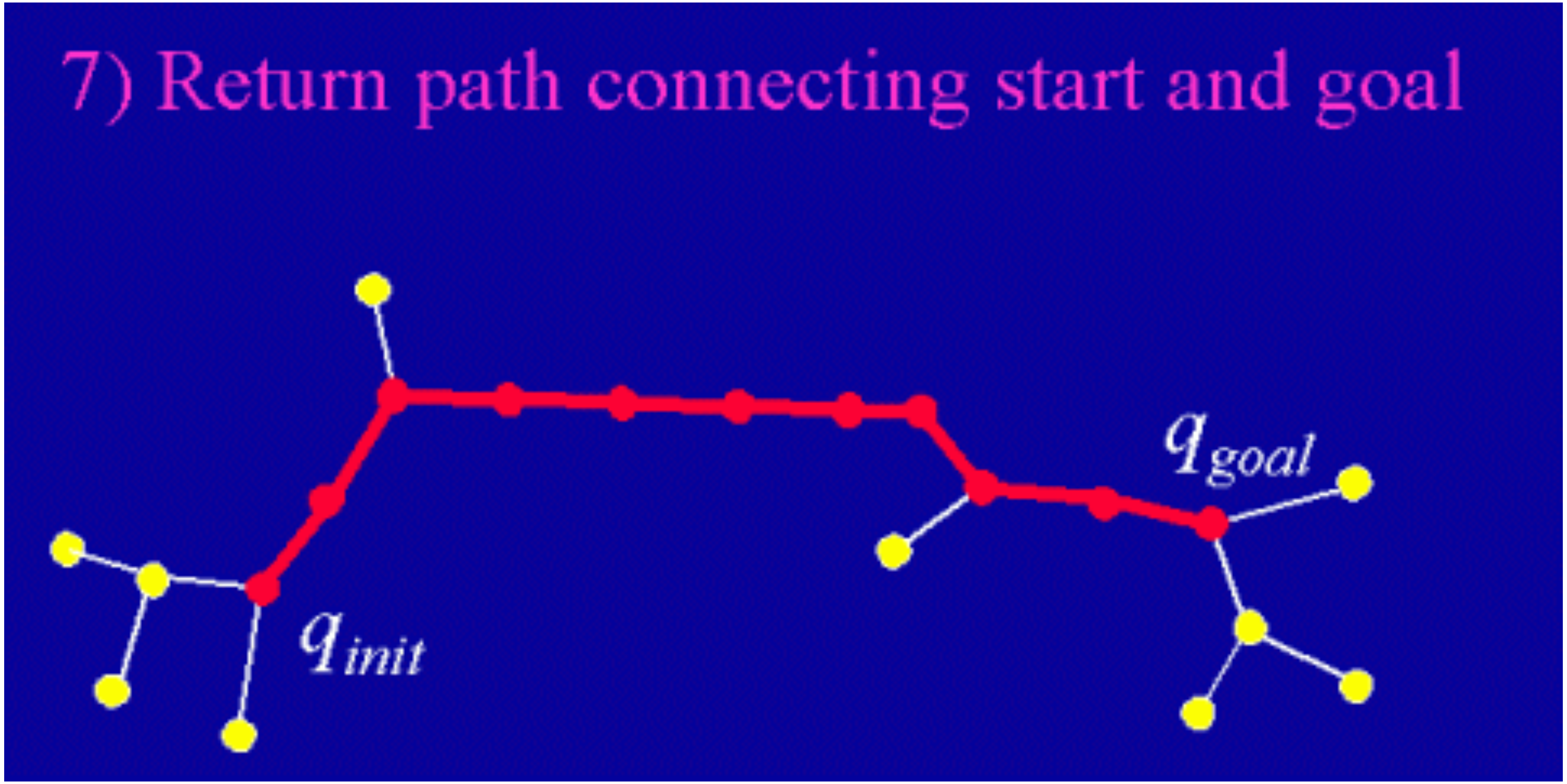
http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation



http://www.kuffner.org/james/plan/algorithm.php

RRT-Connect animation

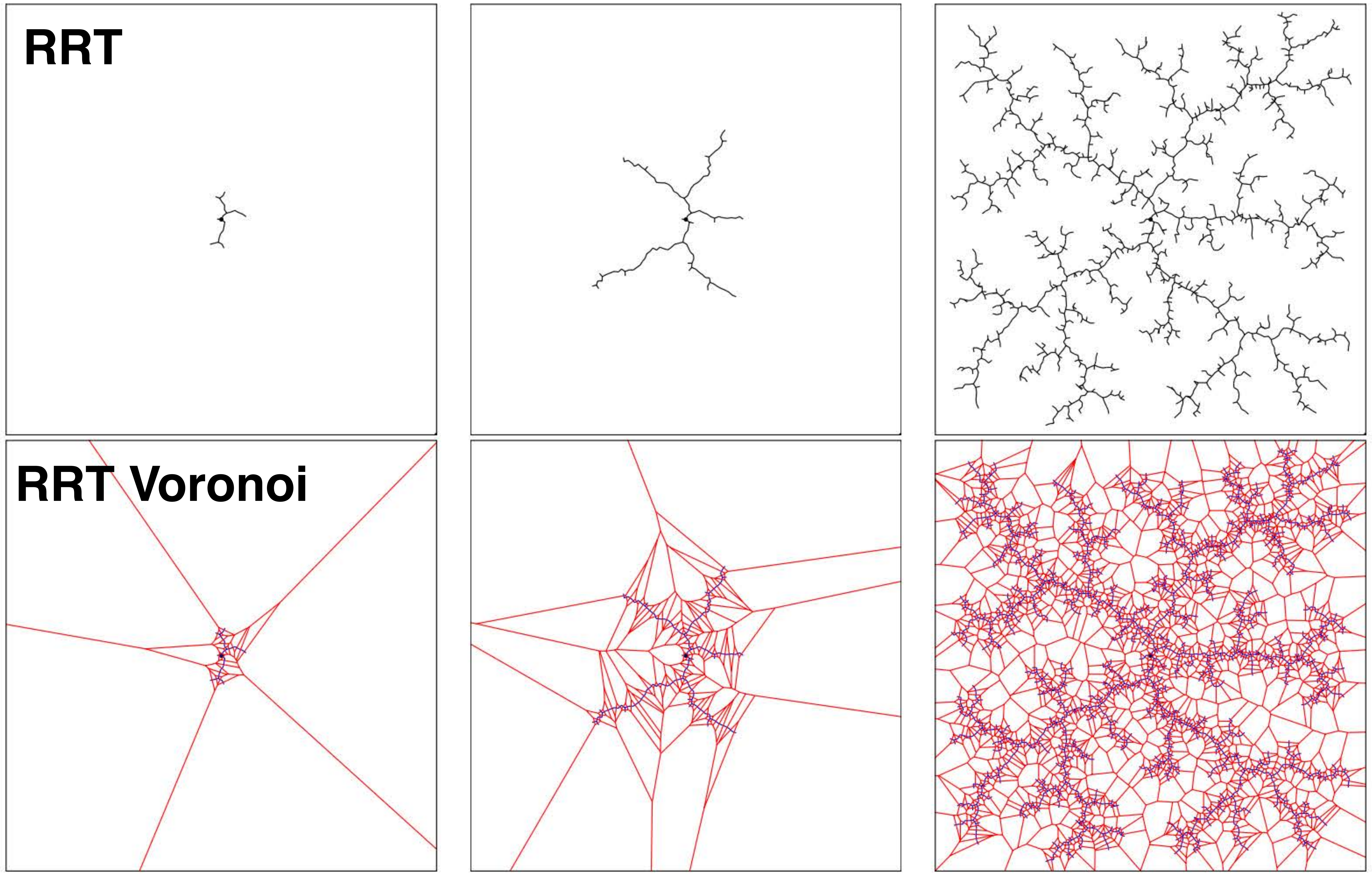


http://www.kuffner.org/james/plan/algorithm.php

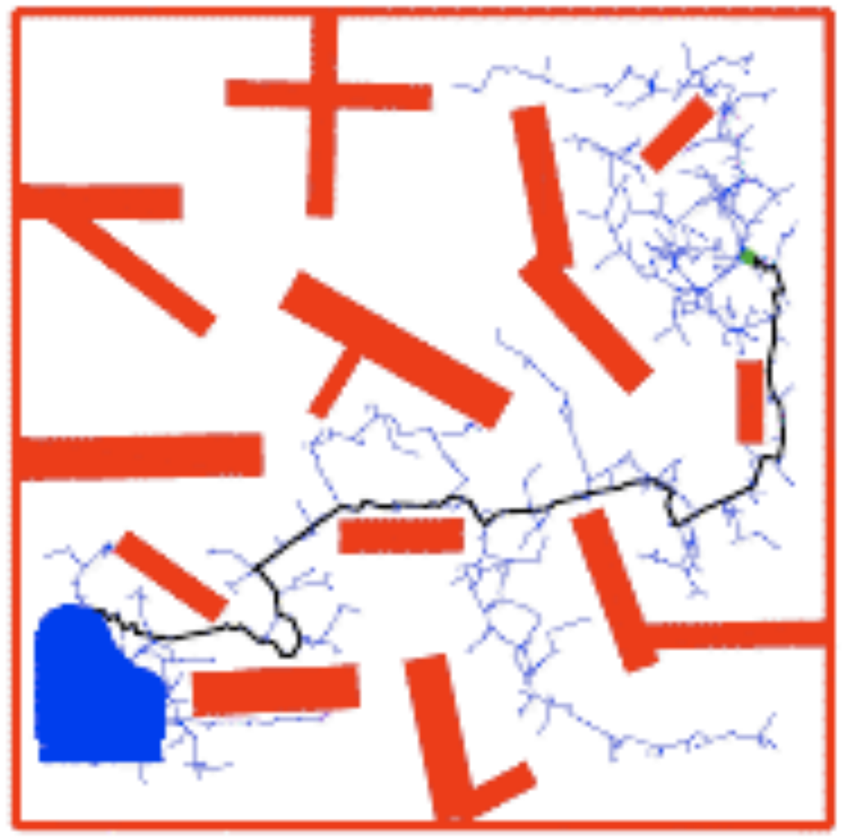
RRT Probabilistic Completeness

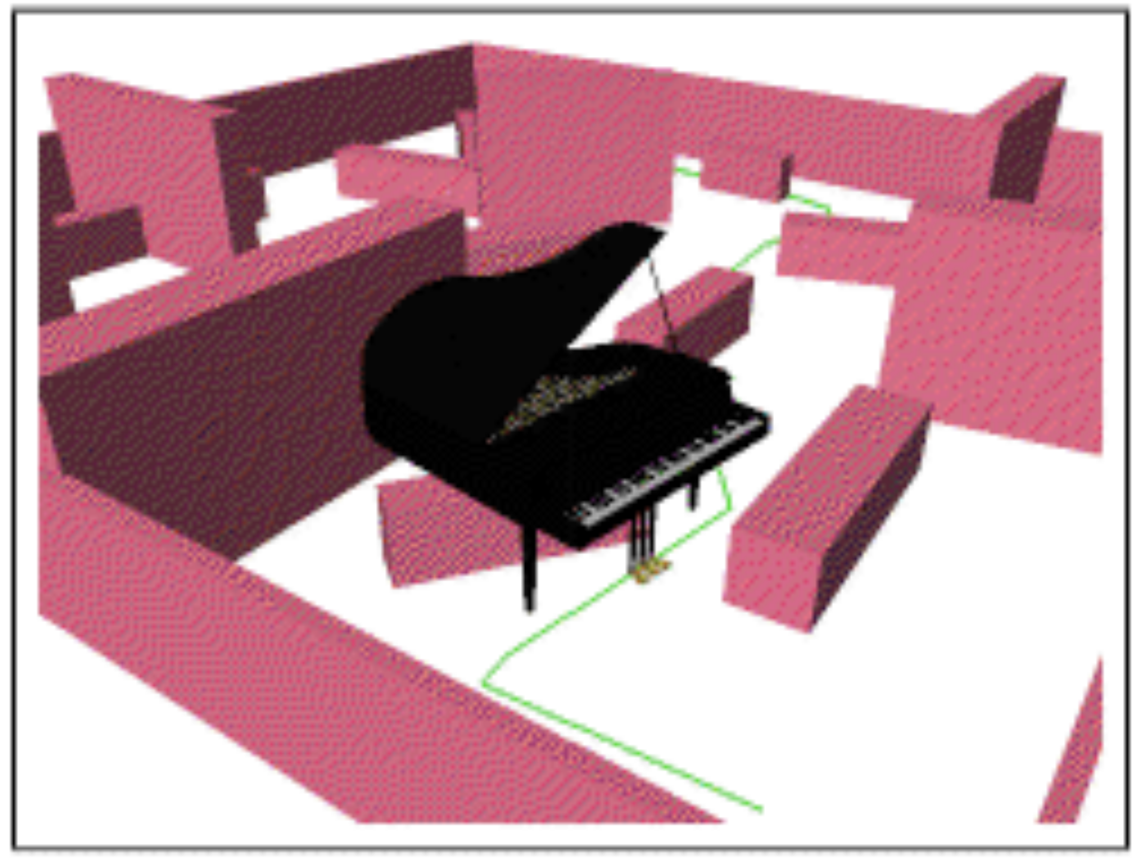
- RRTs converge to a uniform coverage of C-space as the number of samples increases

- Probability a vertex is selected for extension is proportional to its area in Voronoi diagram

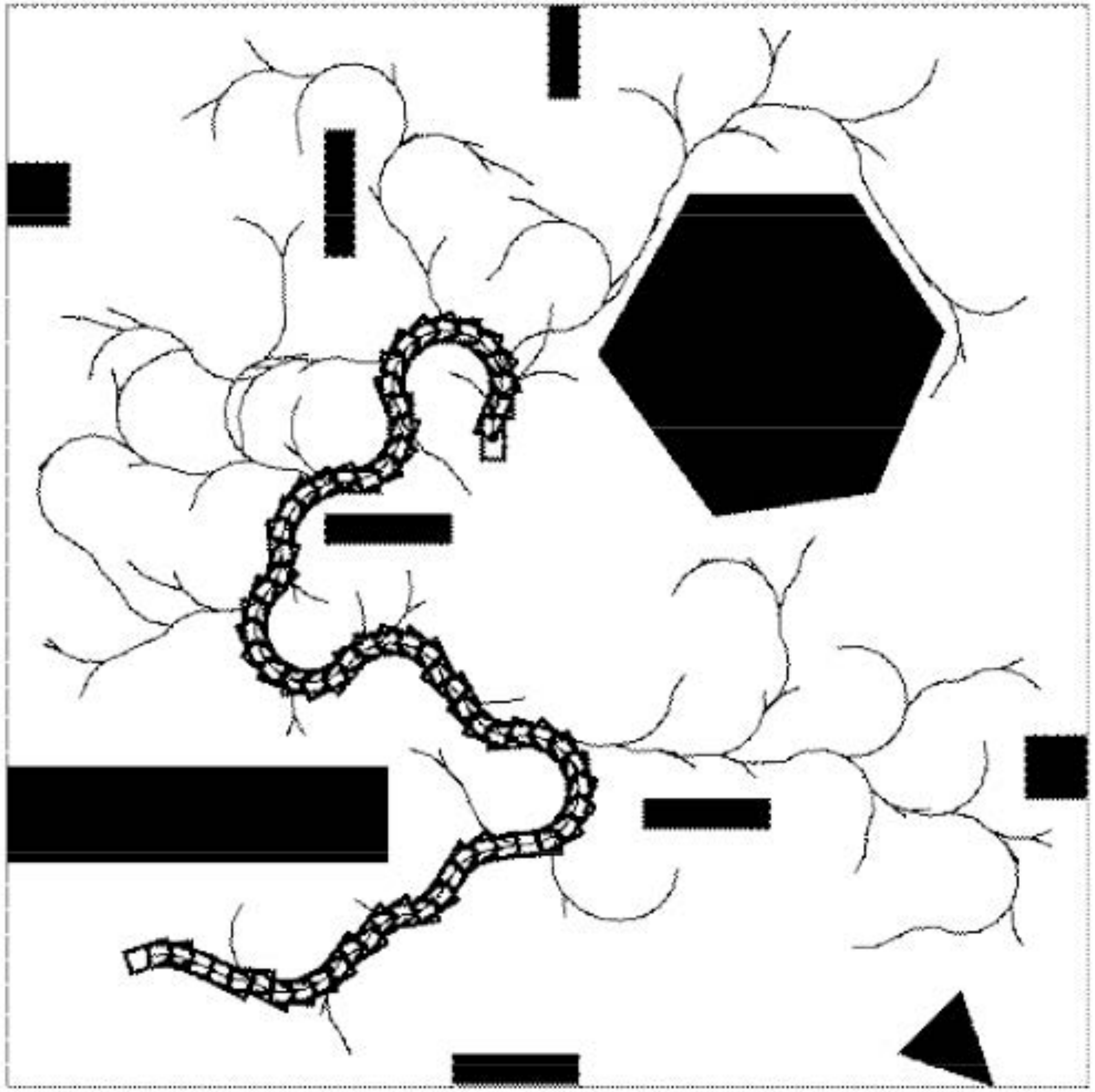


Piano Mover's Problem

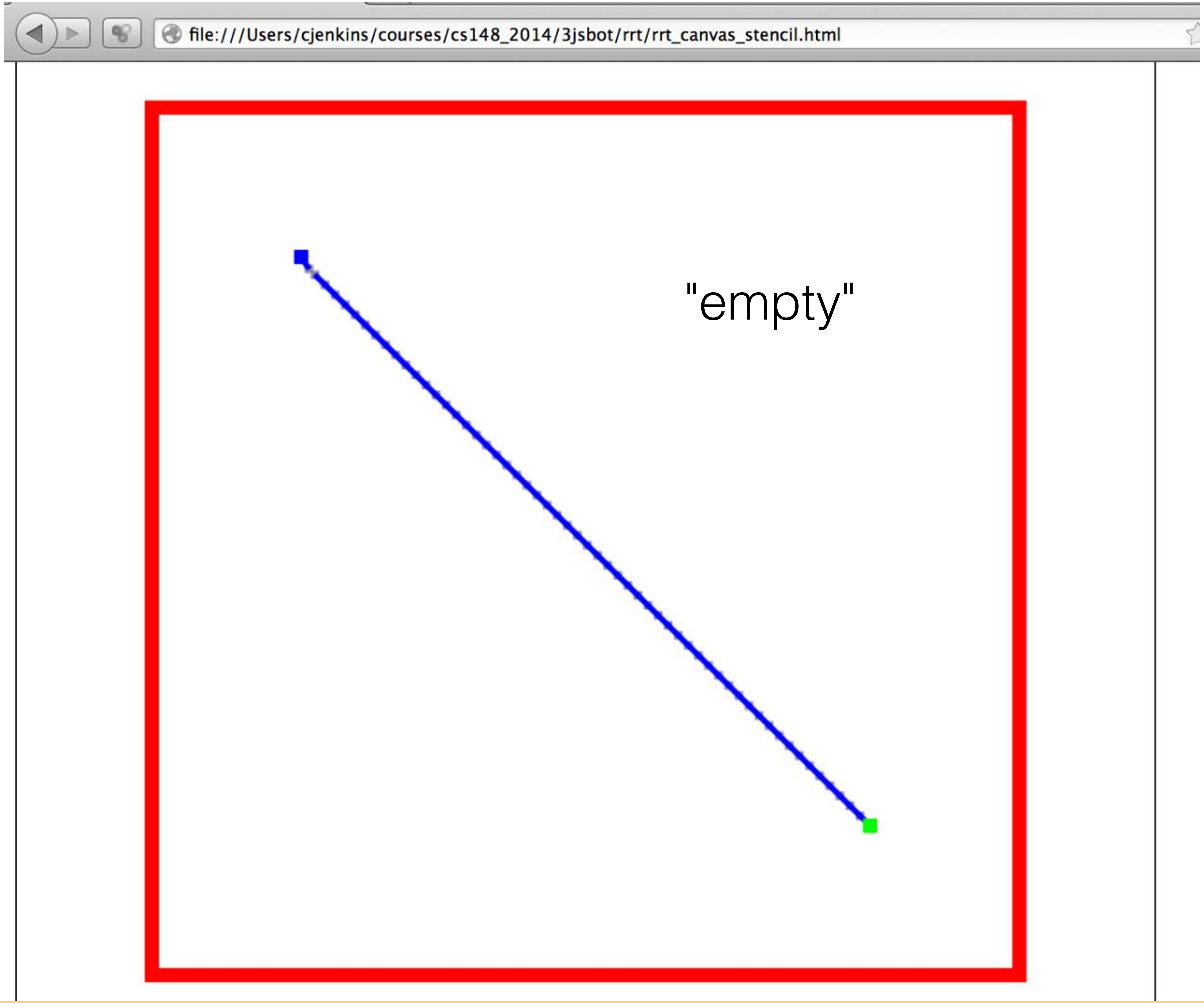


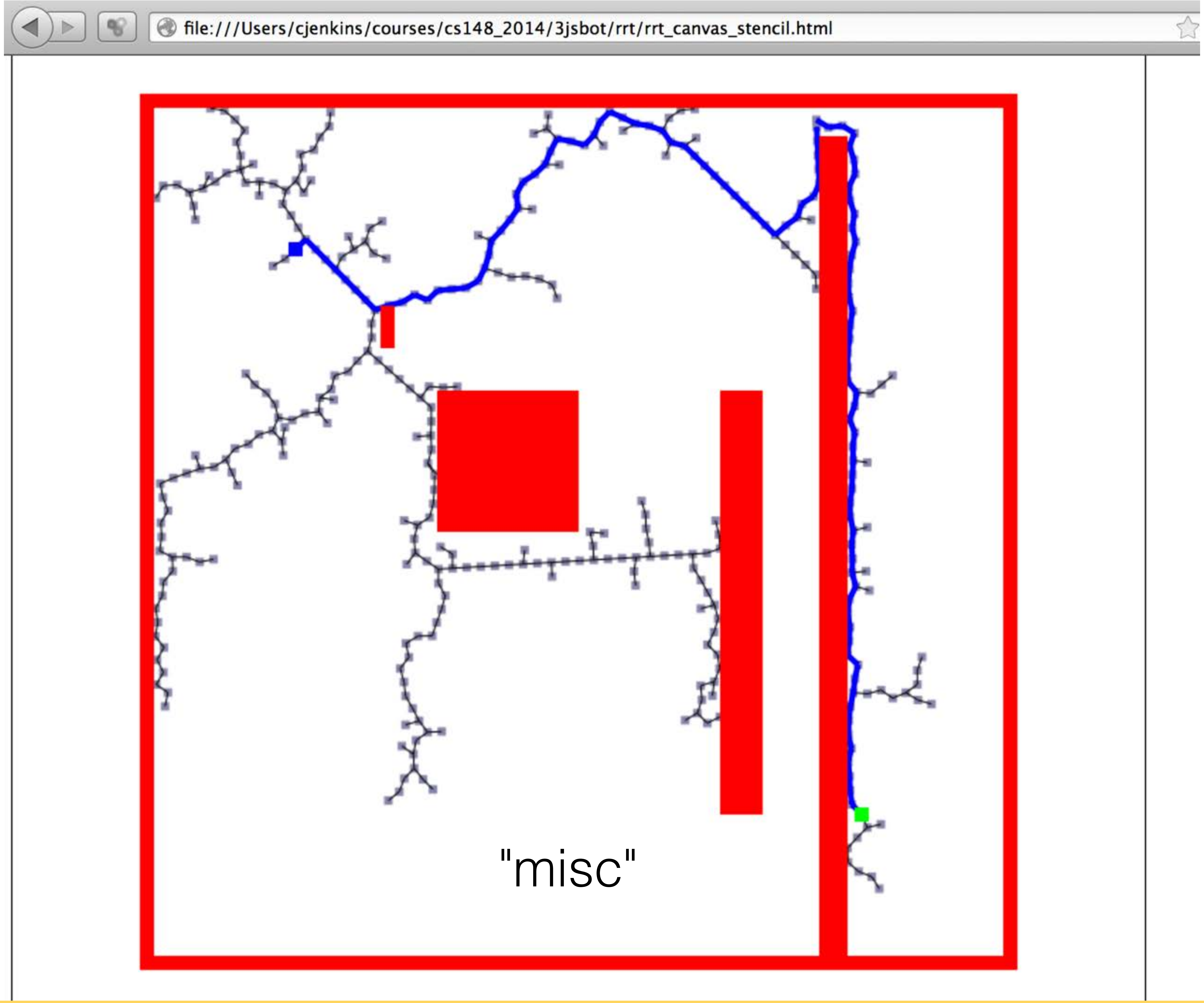


A Car-Like Robot



Canvas Stencil Examples


file:///Users/cjenkins/courses/cs148_2014/3jsbot/rrt/rrt_canvas_stencil.html
"empty"



"misc"

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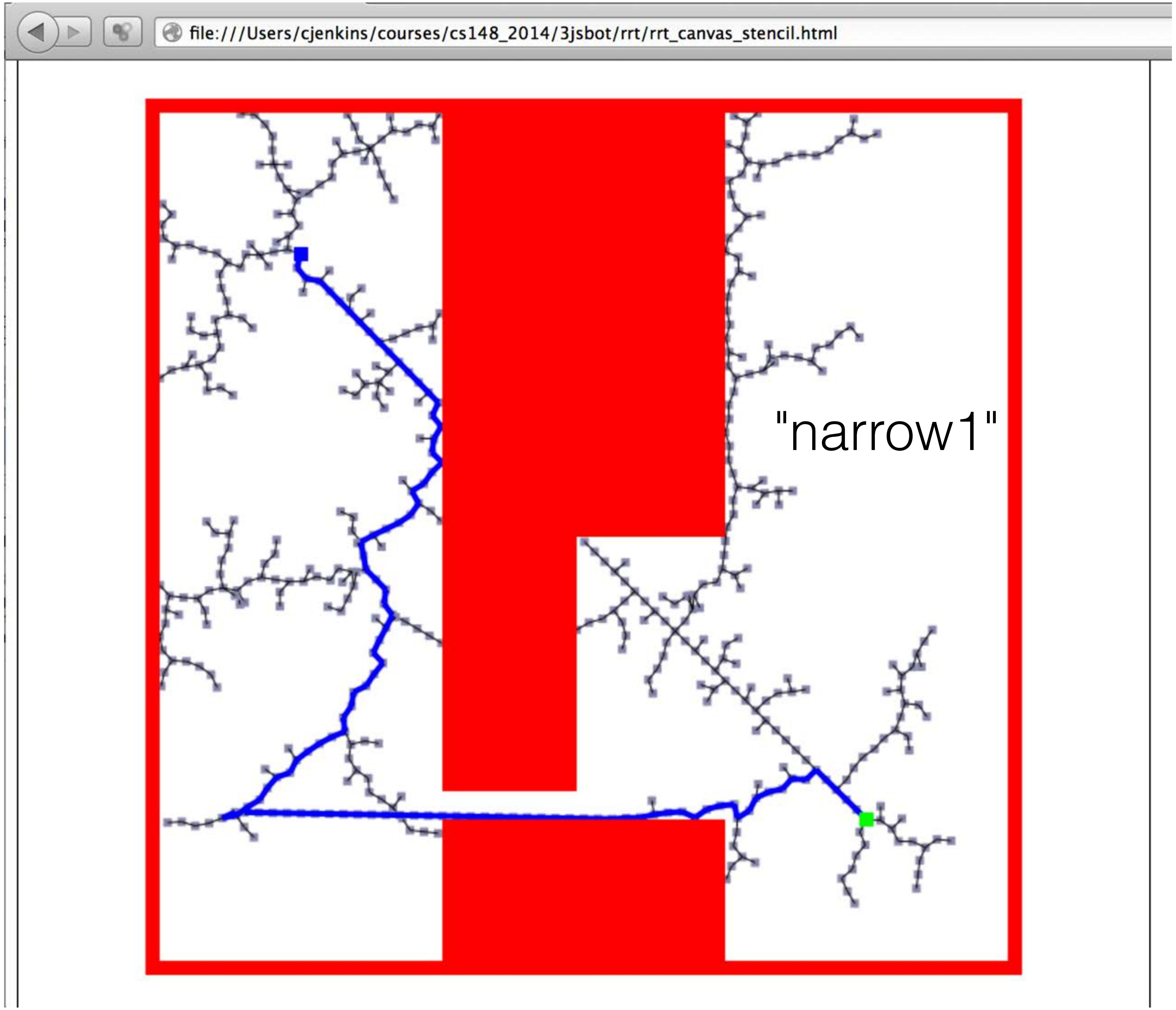
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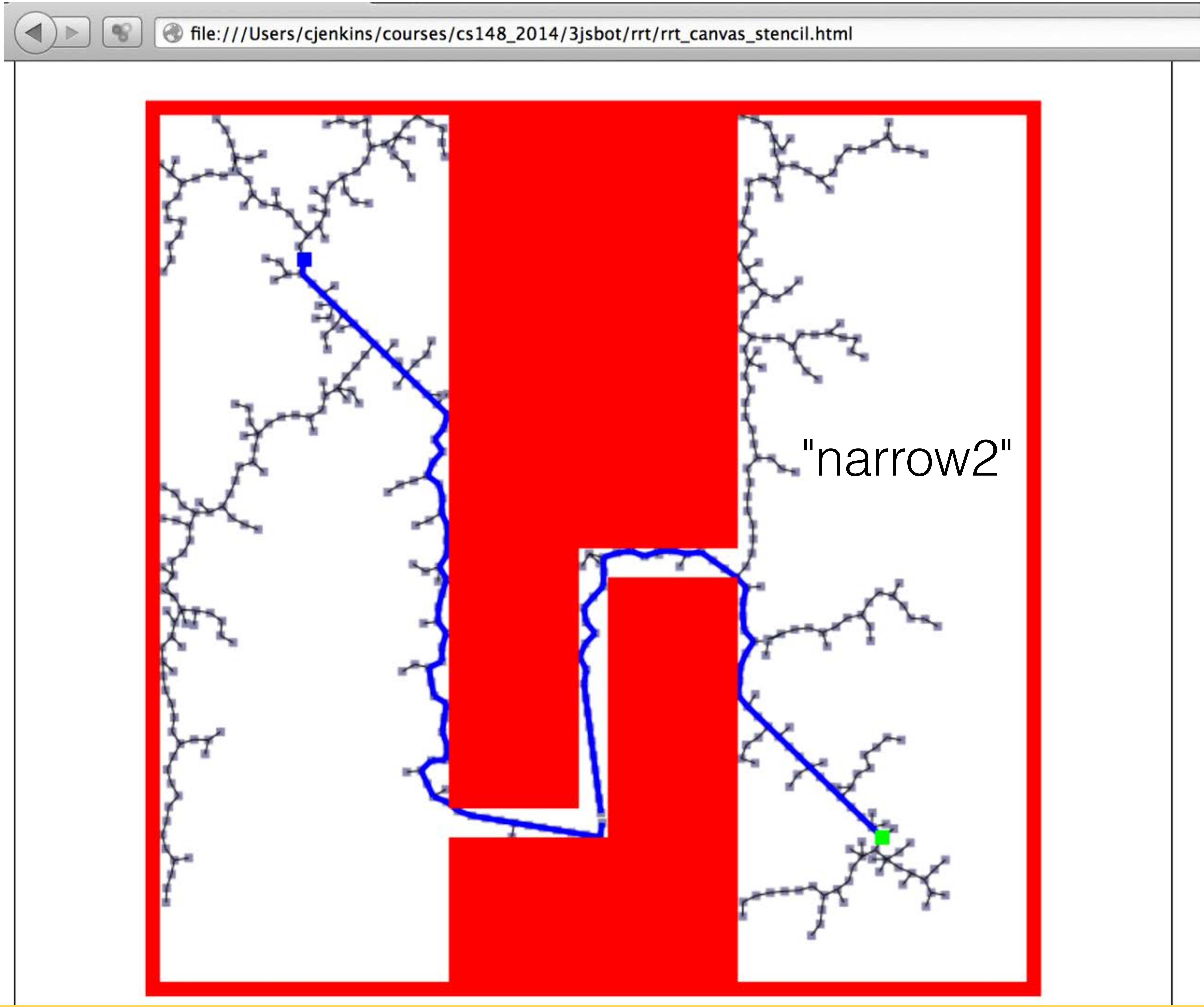
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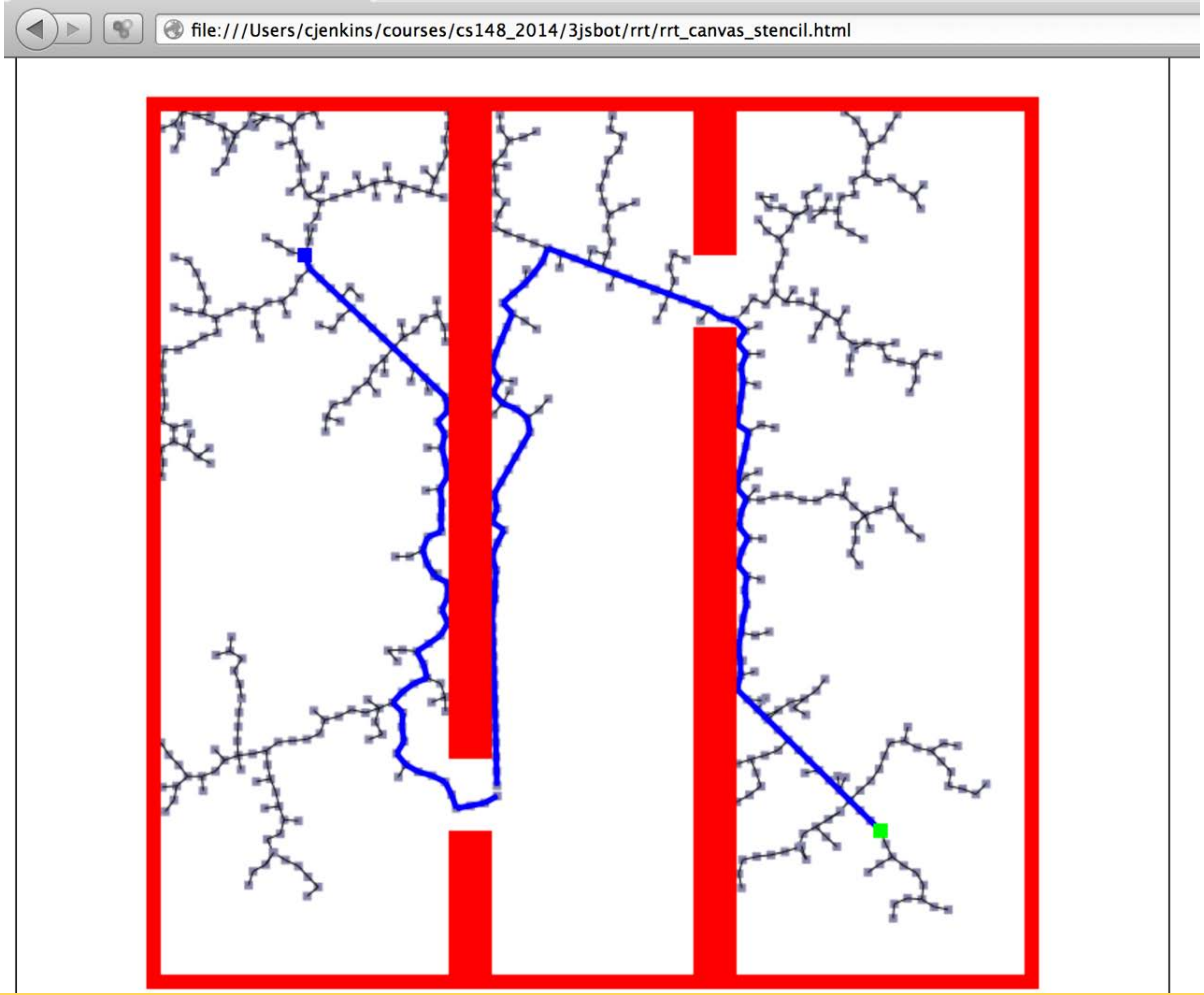
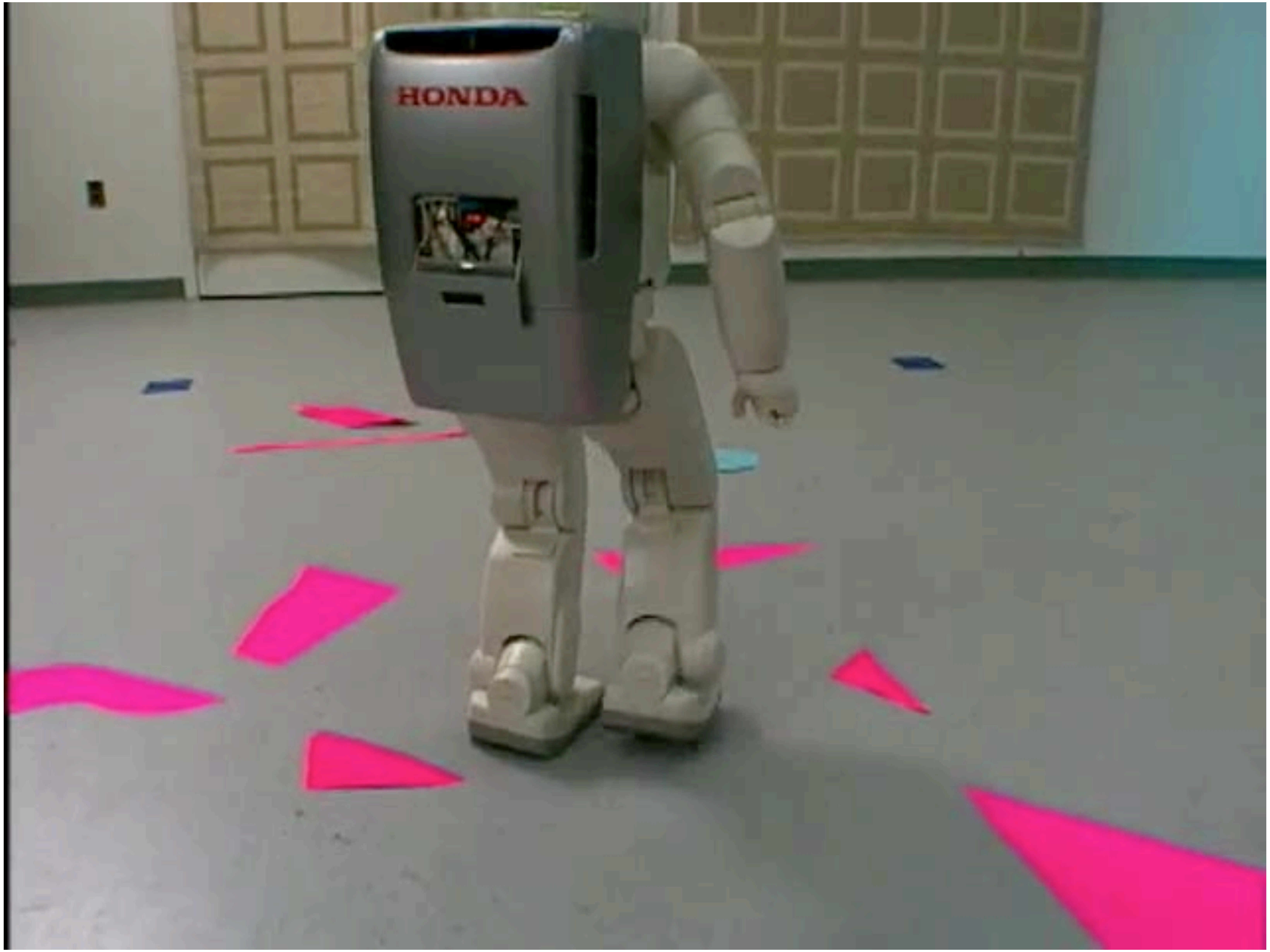


"three_sections"

“We’ve made robot history”



Kuffner/Asimo Discovery Channel feature - https://www.youtube.com/watch?v=wtVmbiTfm0Q

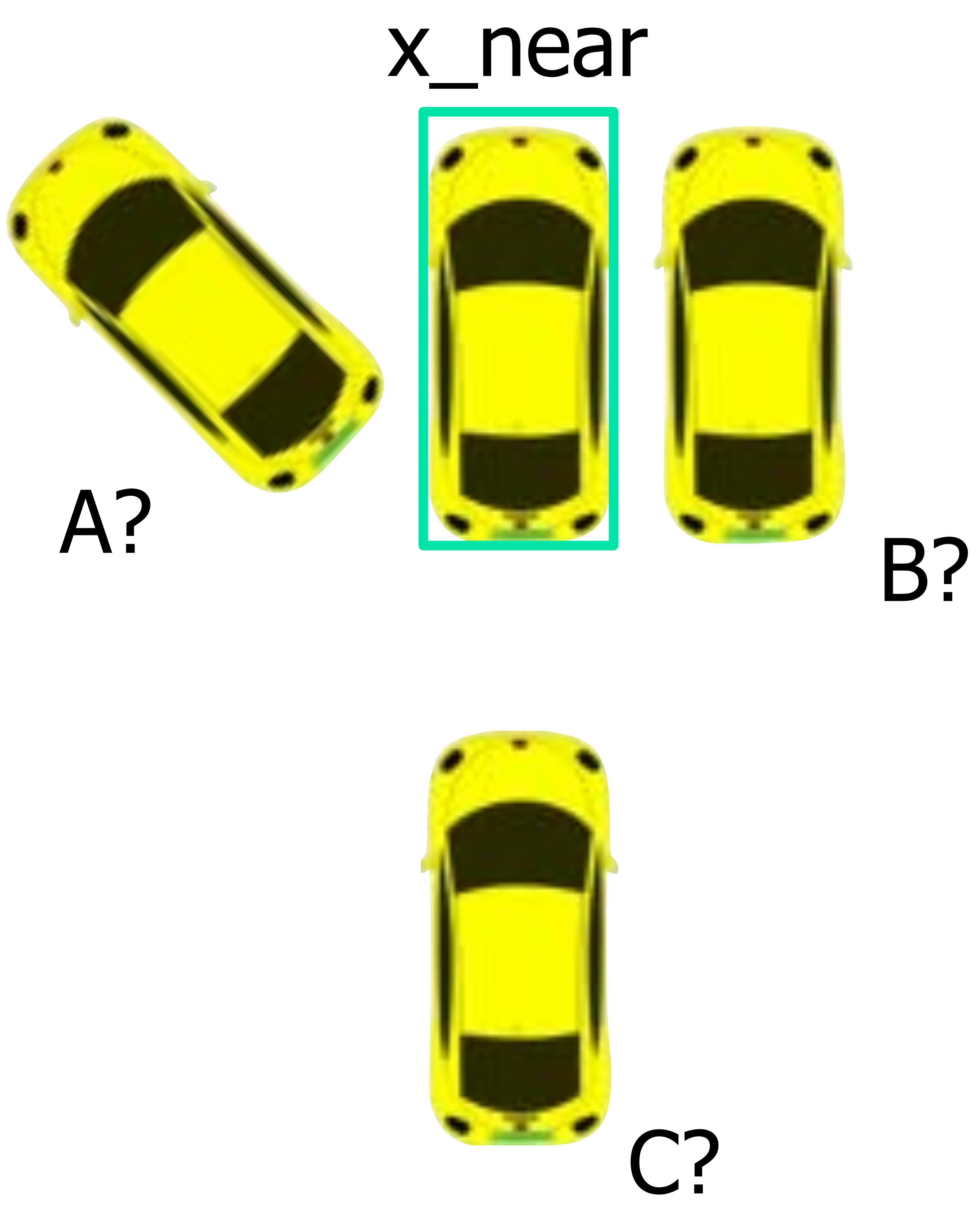
RRT Practicalities

- NEAREST_NEIGHBOR(x_{rand}, T): need to find (approximate) nearest neighbor efficiently
  - KD Trees data structure (upto 20-D) [e.g., FLANN]
  - Locality Sensitive Hashing

- SELECT_INPUT(x_{rand}, x_{near})
  - Two point boundary value problem
    - If too hard to solve, often just select best out of a set of control sequences. This set could be random, or some well chosen set of primitives.

RRT Extension

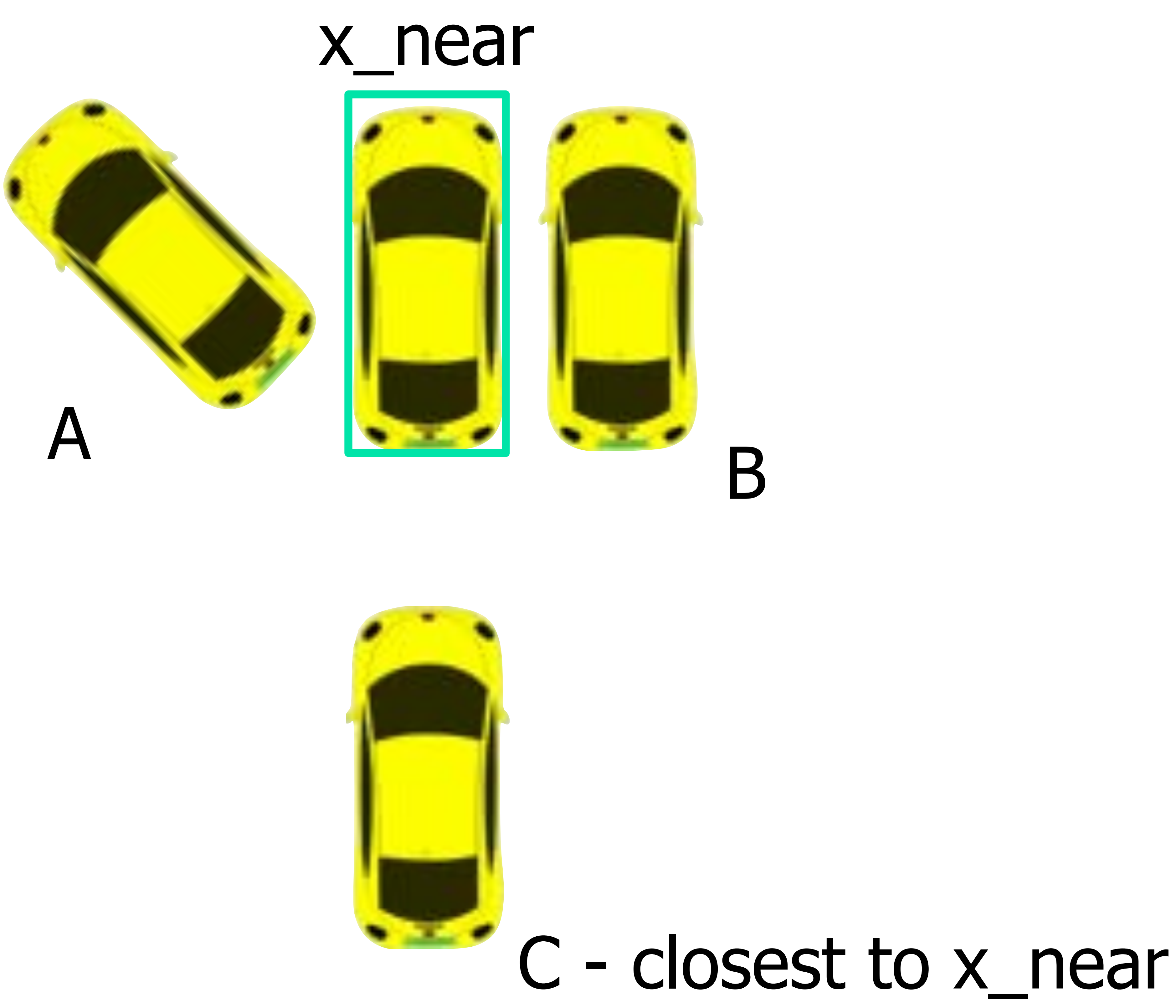
- Non-holonomic: approximately (sometimes as approximate as picking best of a few random control sequences) solve two-point boundary value problem



Slide borrowed from Dieter Fox

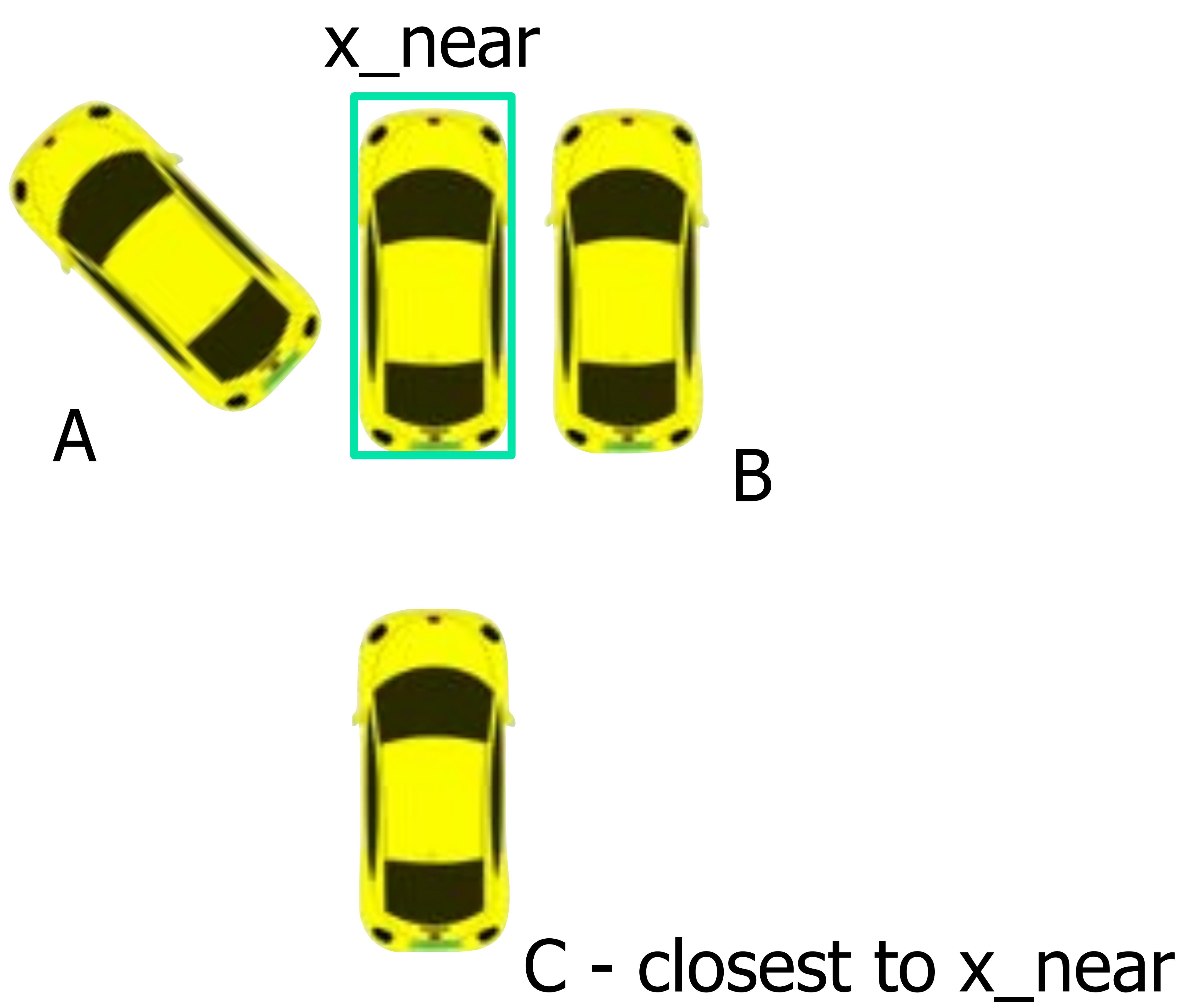
RRT Extension

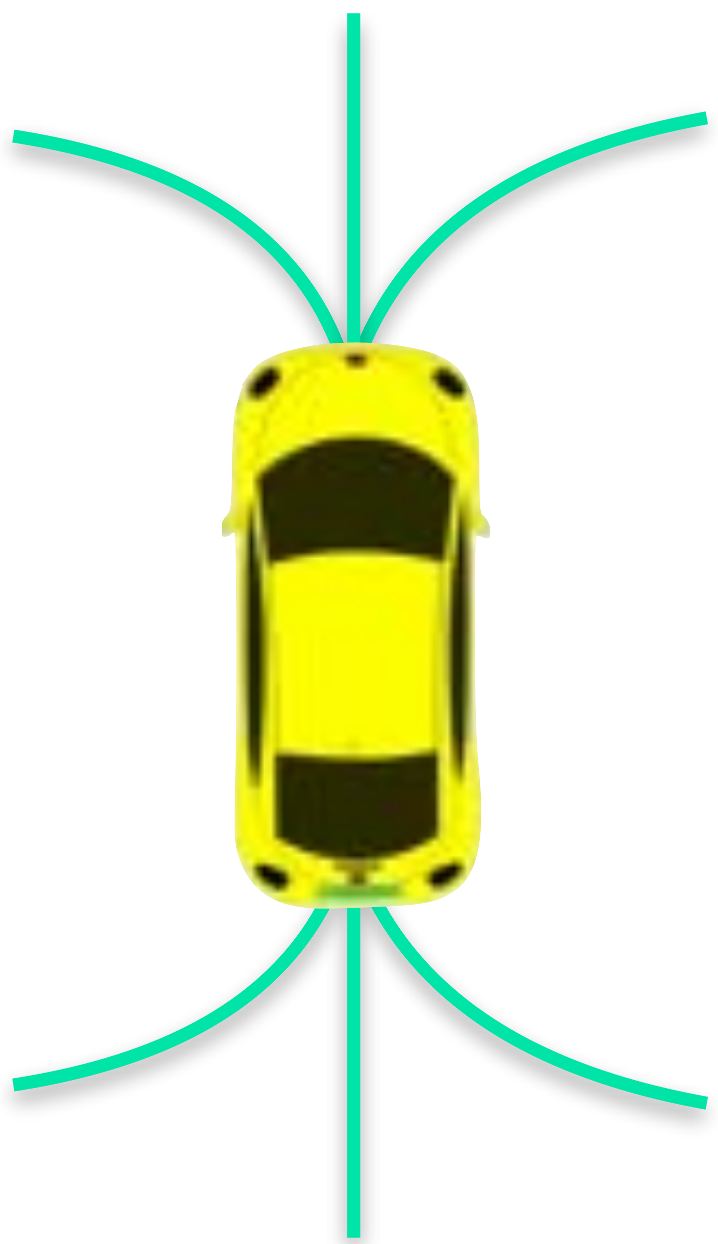
- Non-holonomic: approximately (sometimes as approximate as picking best of a few random control sequences) solve two-point boundary value problem



RRT Extension

- Non-holonomic: approximately (sometimes as approximate as picking best of a few random control sequences) solve two-point boundary value problem

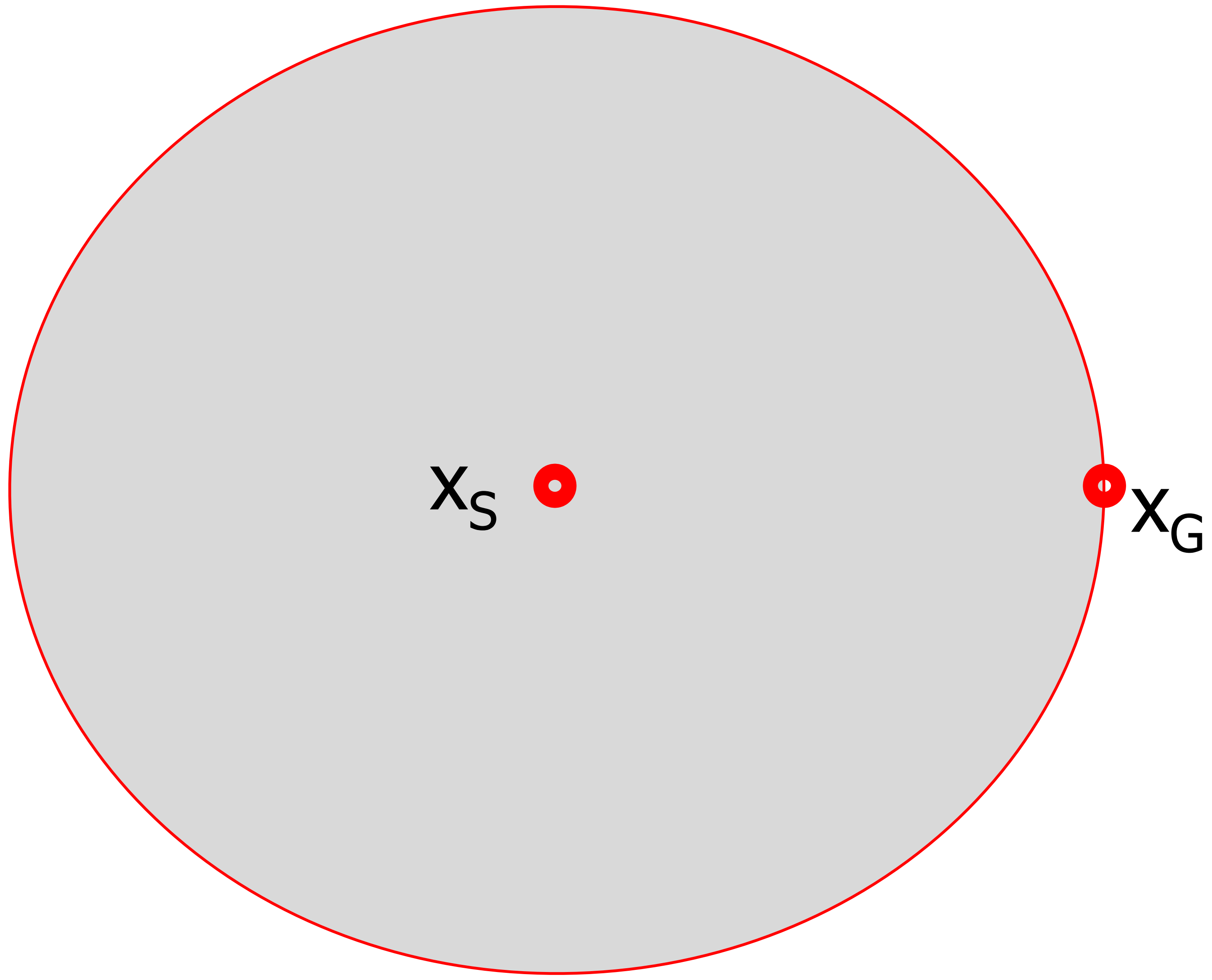




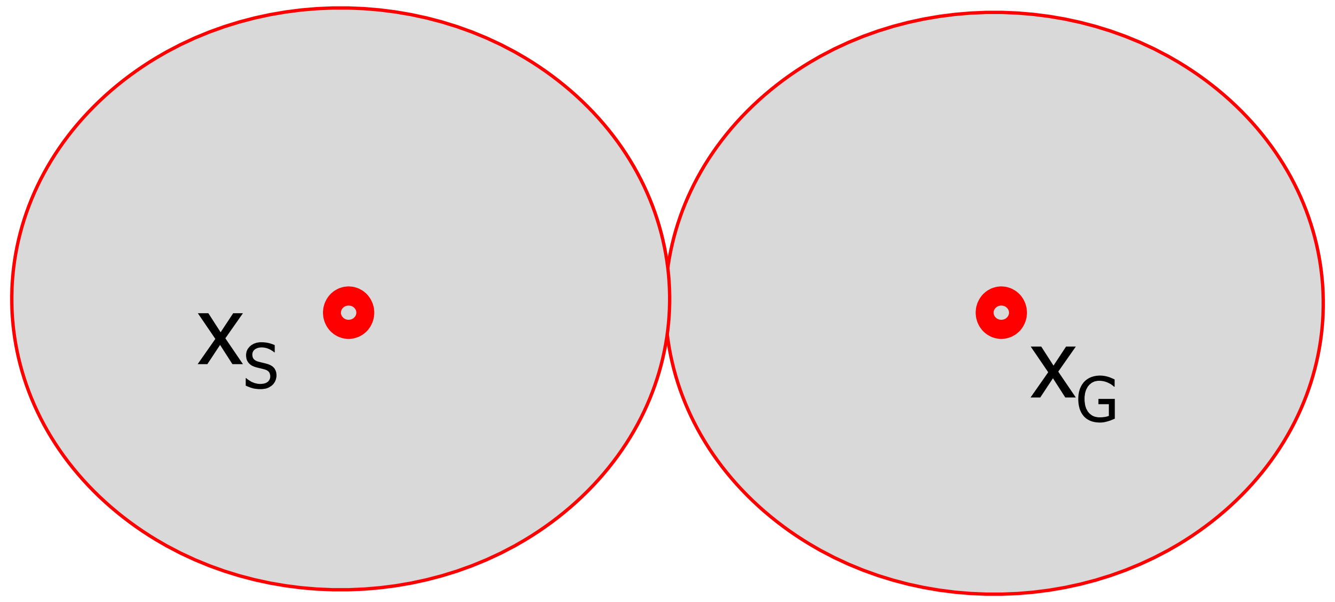
Slide borrowed from Dieter Fox

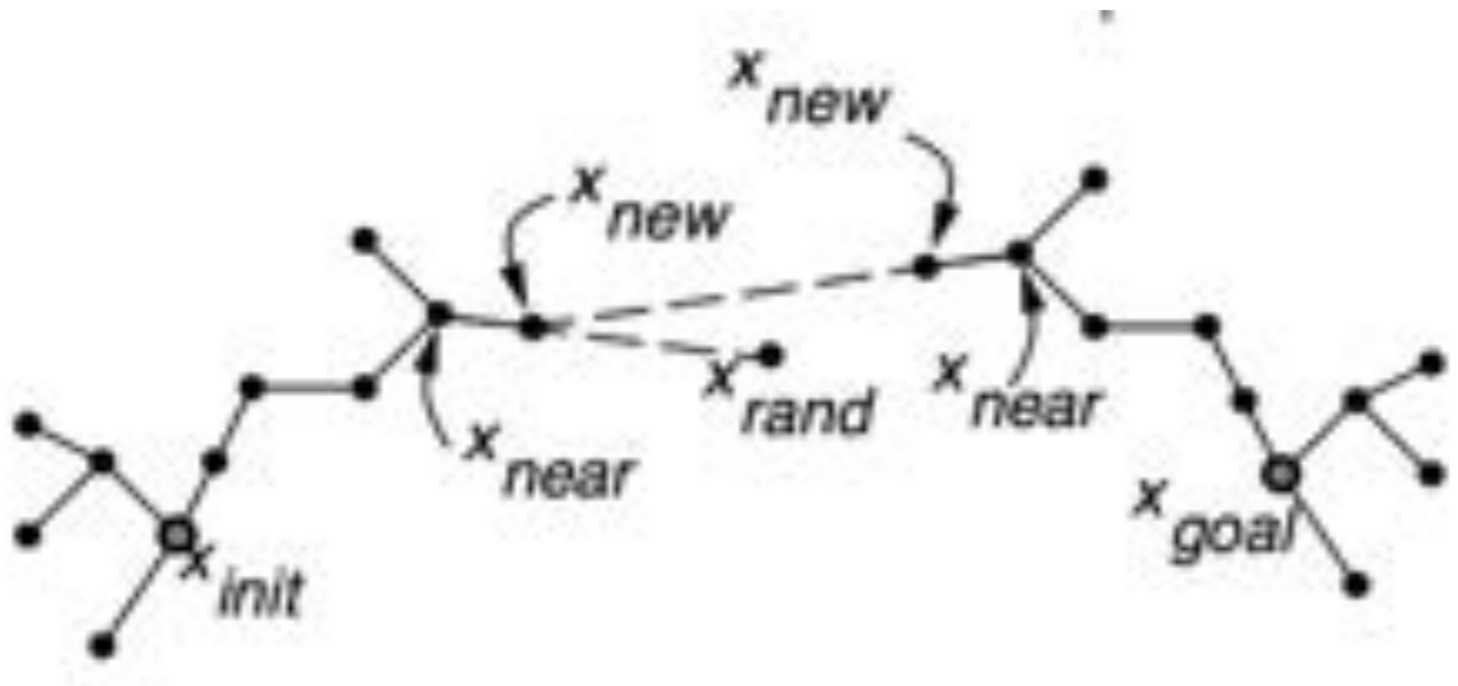
Bi-directional RRT

- Volume swept out by unidirectional RRT:



- Volume swept out by bi-directional RRT:





- Difference more and more pronounced as dimensionality increases

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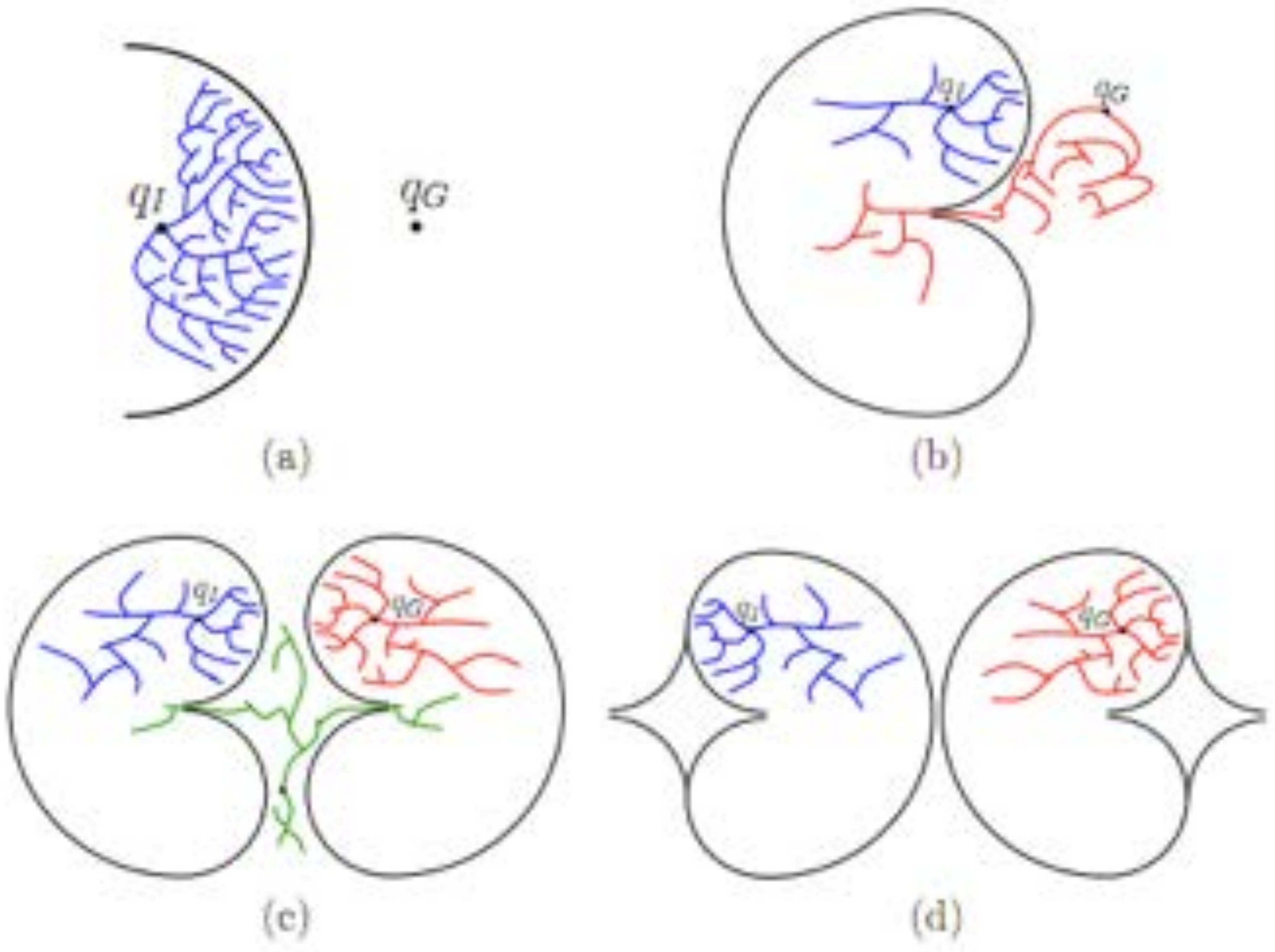
Multi-directional RRT

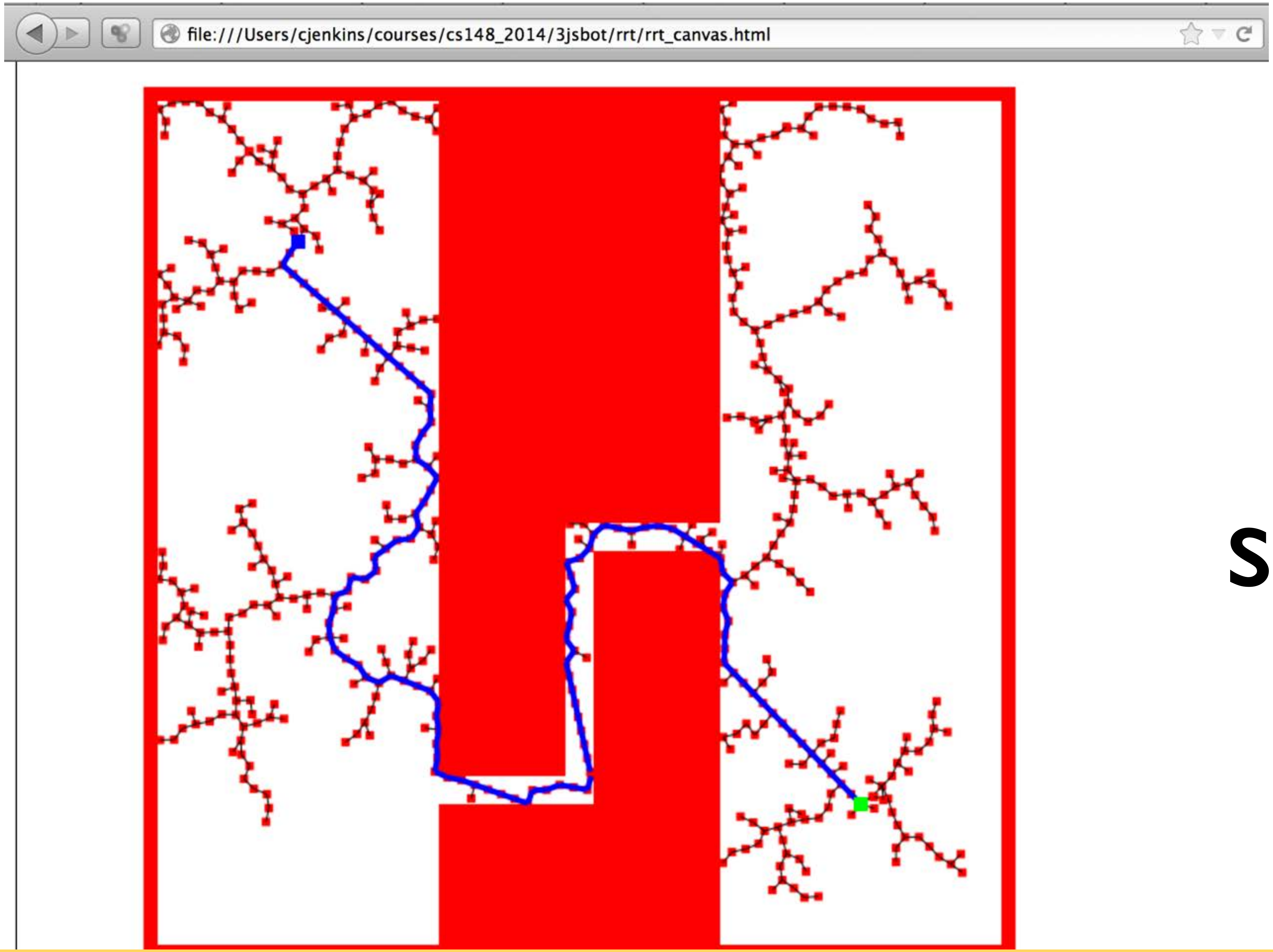
- Planning around obstacles or through narrow passages can often be easier in one direction than the other



(a) (b) (c) (d)

Slide borrowed from Dieter Fox

RRTs can take a lot of time...



Is there a simpler way?

Next Lecture

Planning - V - Collision Detection