

# Lecture 03

## Linear Algebra Refresher



# Course Logistics

- OH Starting this week! Details on the course webpage.
- Everyone should be on Ed discussion board now.
- Everyone should be on Gradescope now.
- **Quiz 1 was posted yesterday and was due today at noon.**
- Project 1 will be posted today 01/29 and will be due 02/05.
- **Action items for you:**
  - Announcements are done via Ed discussion. So make sure you are getting email notifications from Ed.

## Instructors



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## Teaching Assistants



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**Office Hours:** Wednesdays and Thursdays 3:00-4:00 PM CT at Keller 2-209

**For UNITE section** the office hours will be on demand. Send us an email and we will schedule a zoom call.

# FAQs

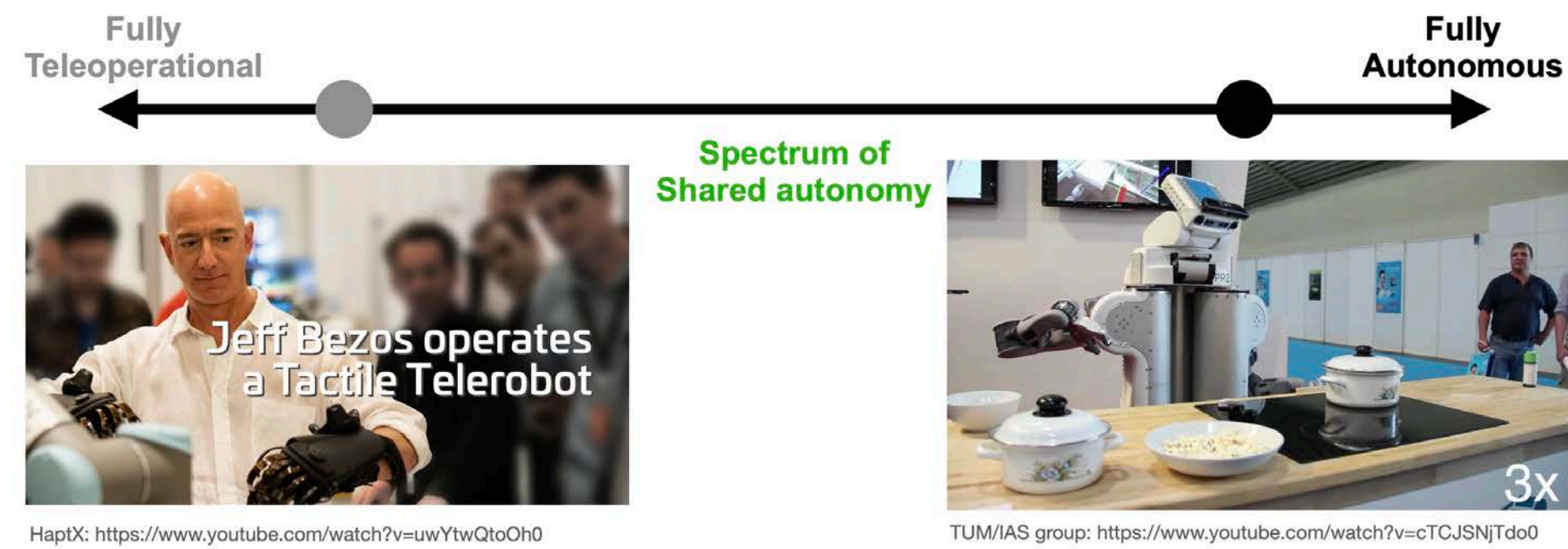
- **When will the slides be posted?**
  - Slides for the lecture will be posted on the course webpage by before every lecture.
- **How much programming knowledge is required for this course?**
  - You will use JavaScript for the most part.
  - Proficiency in any programming language (Python, C, C++, Java, JavaScript...) is essential to perform in the course.
  - Being able to debug your program effectively will be the key.
- **Do I need to know ROS?**
  - No!
- **Will I learn ROS?**
  - You will be exposed to ROS in P7.
- **Any tips to succeed in this course?**
  - Start the projects as and when they are released.
  - Use office hours and Ed discussion board.





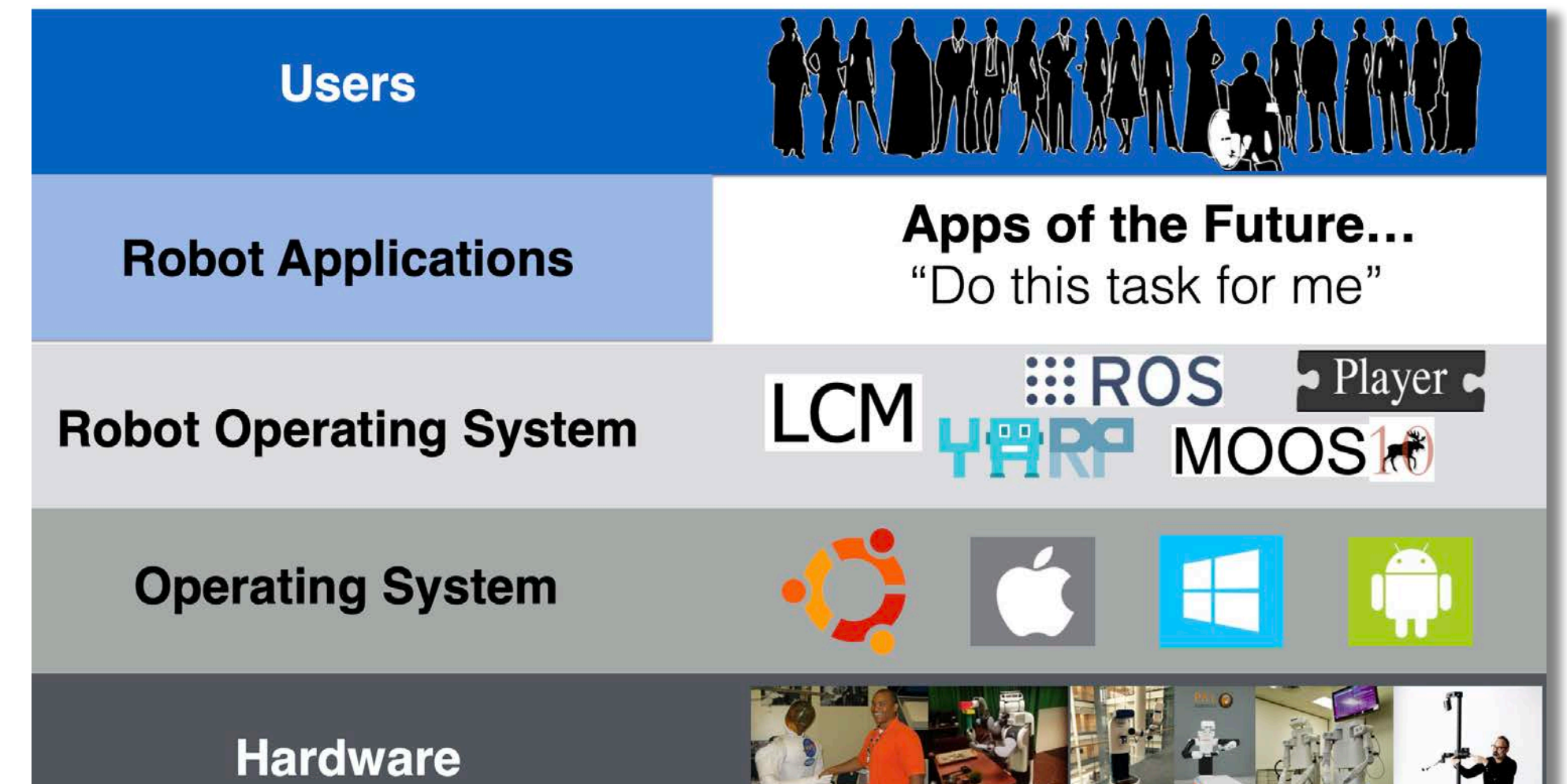
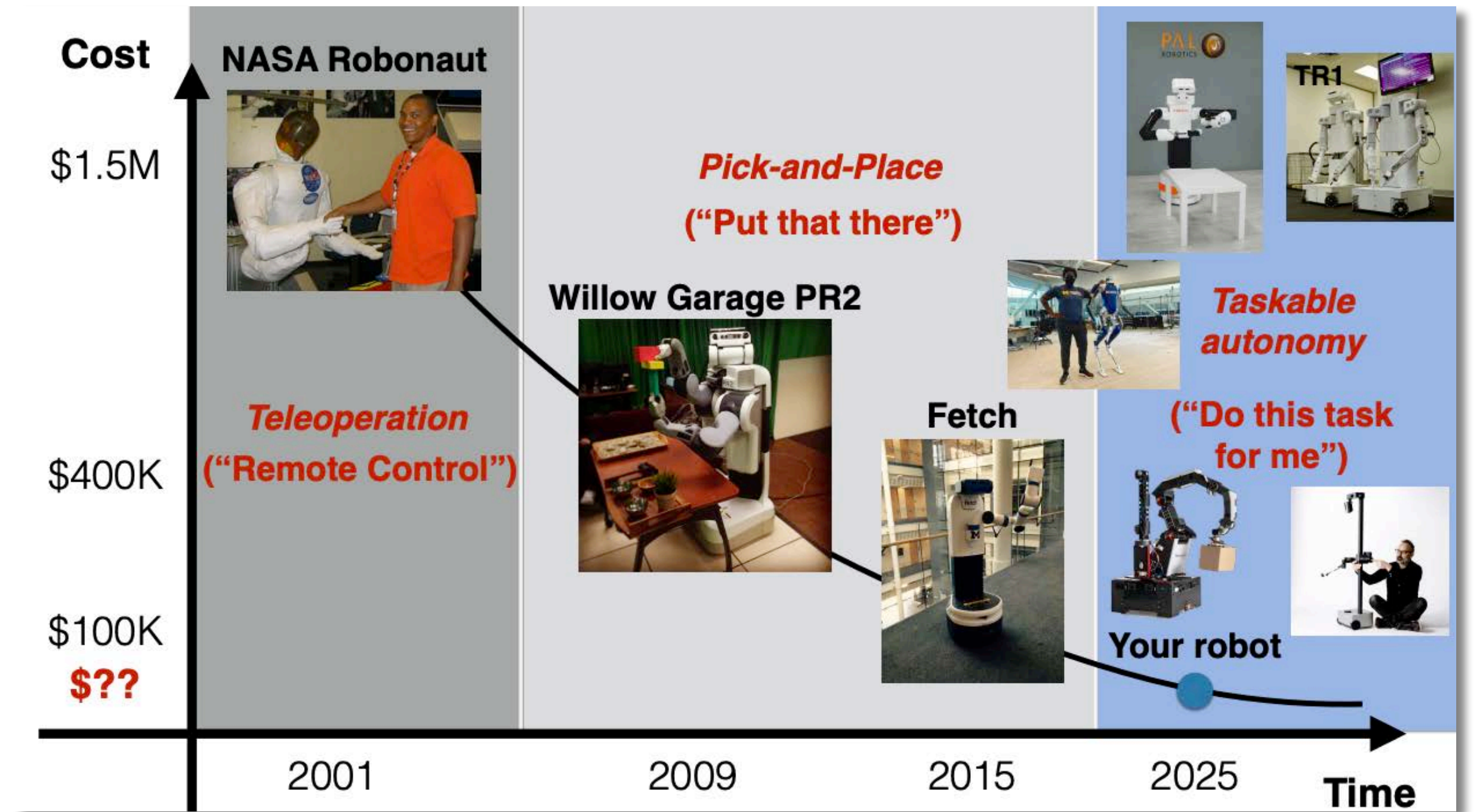
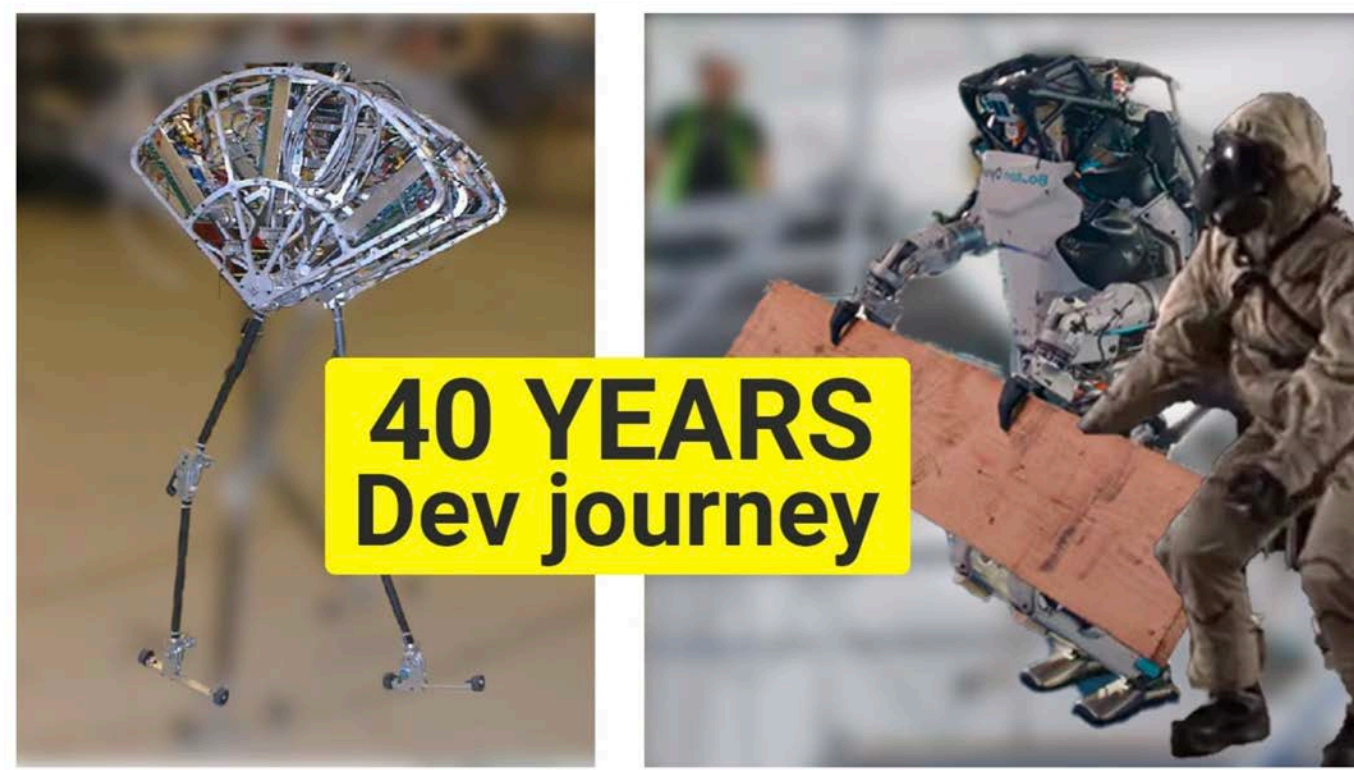
# Previously

What are intelligent robotic systems?

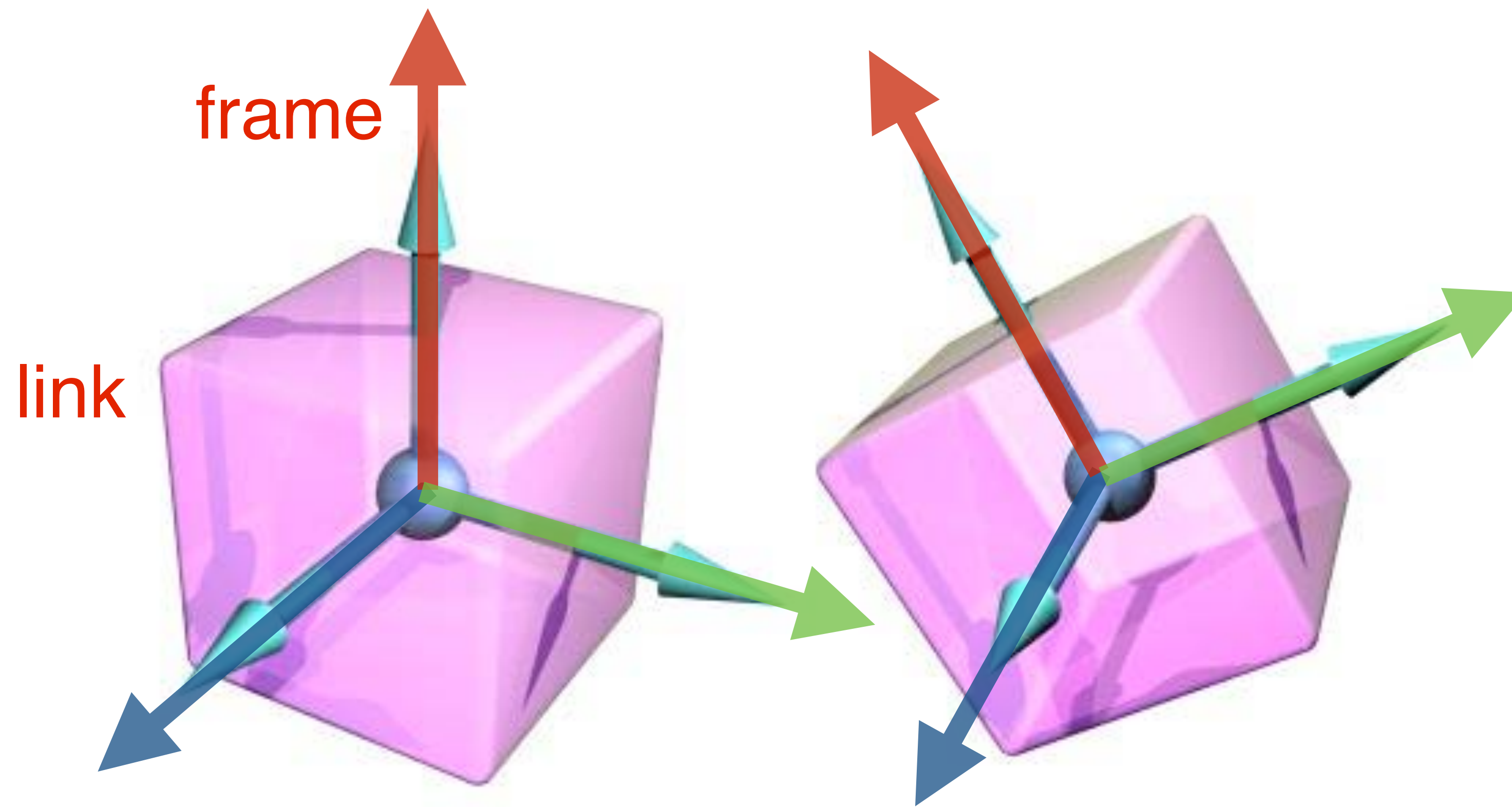


...systems that can perform **Sense-Plan-Act...**

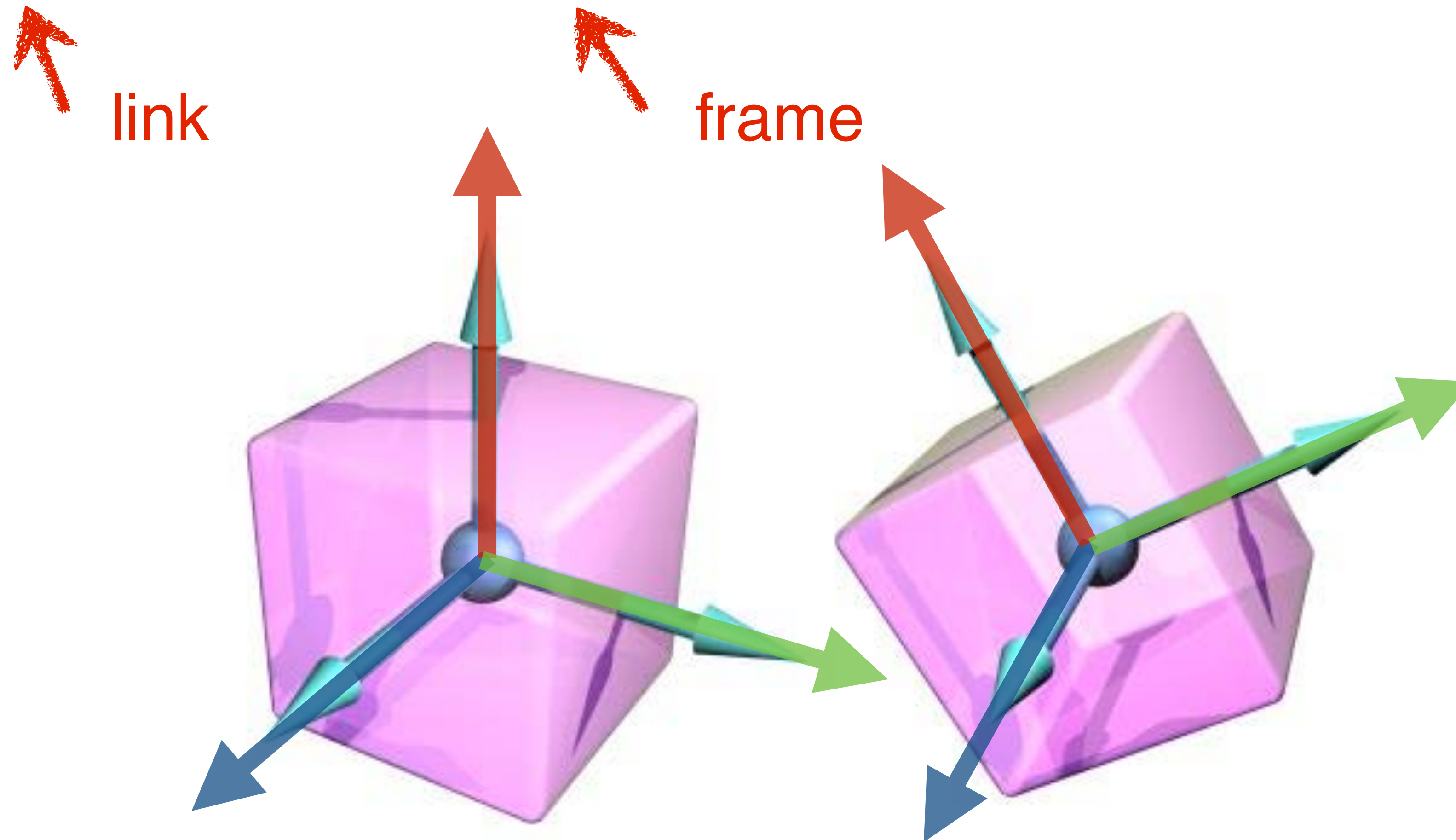
... learn skills ... transfer these skills ... adapt to new environments ... work with humans ...







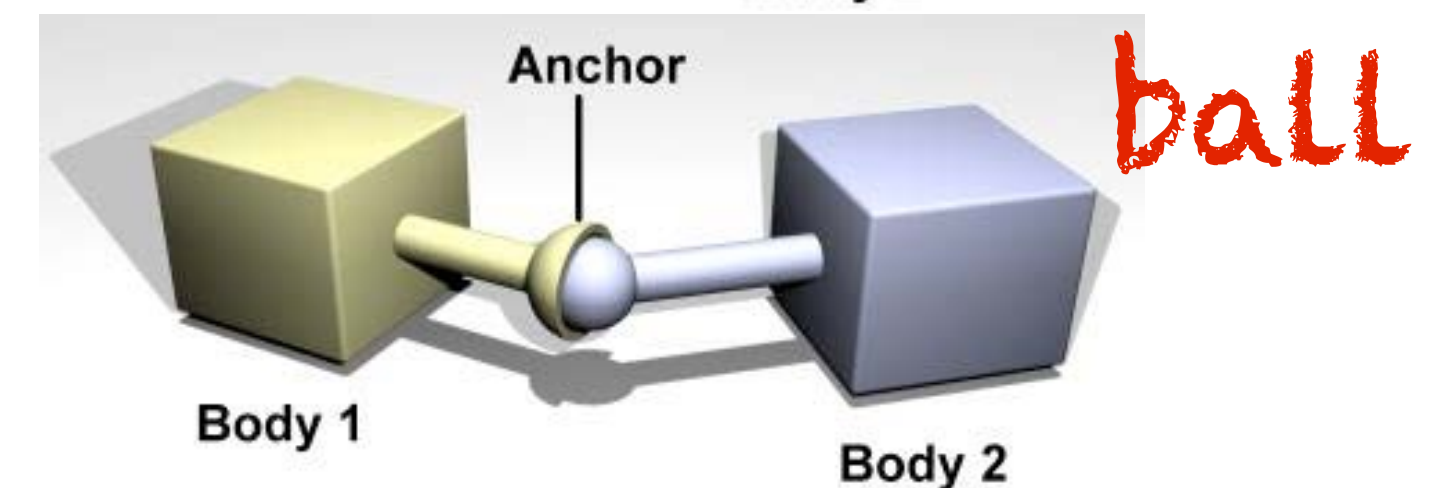
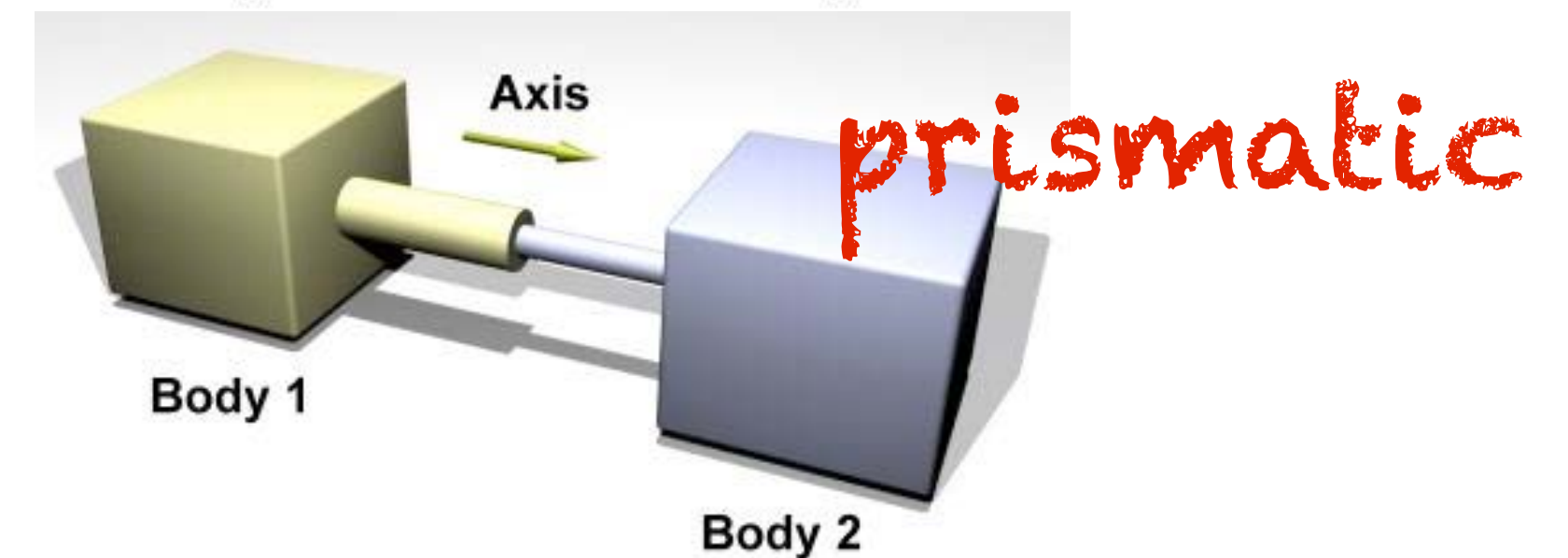
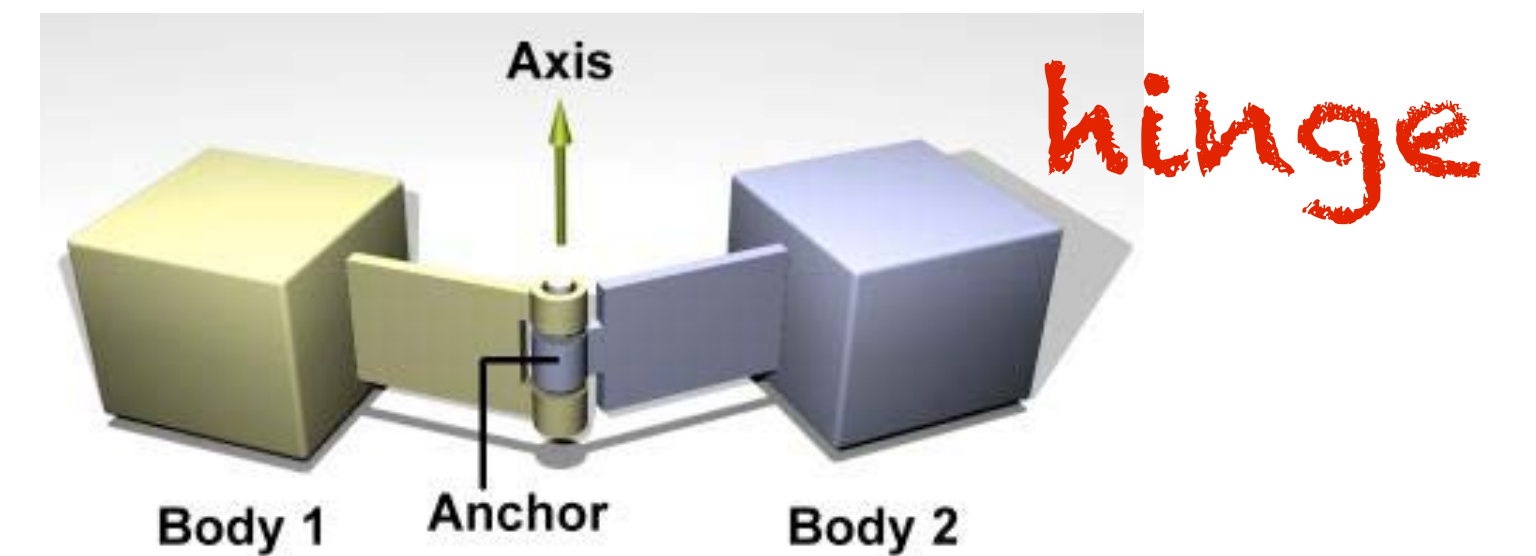
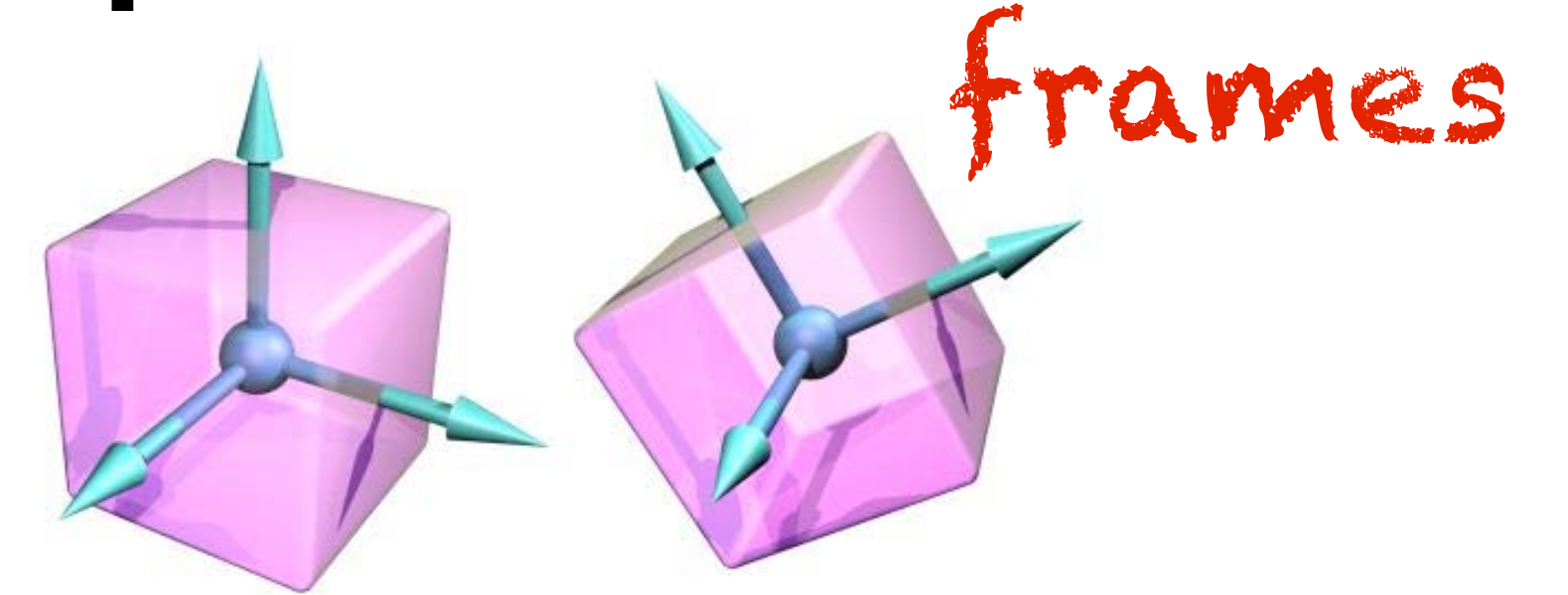
- Each body has its own coordinate system





# DOFs and Coordinate Spaces

- Each body has its own frame
  - Joints connect two links (rigid bodies)
    - e.g., hinge, prismatic, ball-socket
  - A motor exerts force on a DOF axis
- ← Linear algebra
- Matrix transformations used to relate coordinate systems of bodies and joints
  - Spatial geometry attached to each link, but does not affect the body's coordinate frame

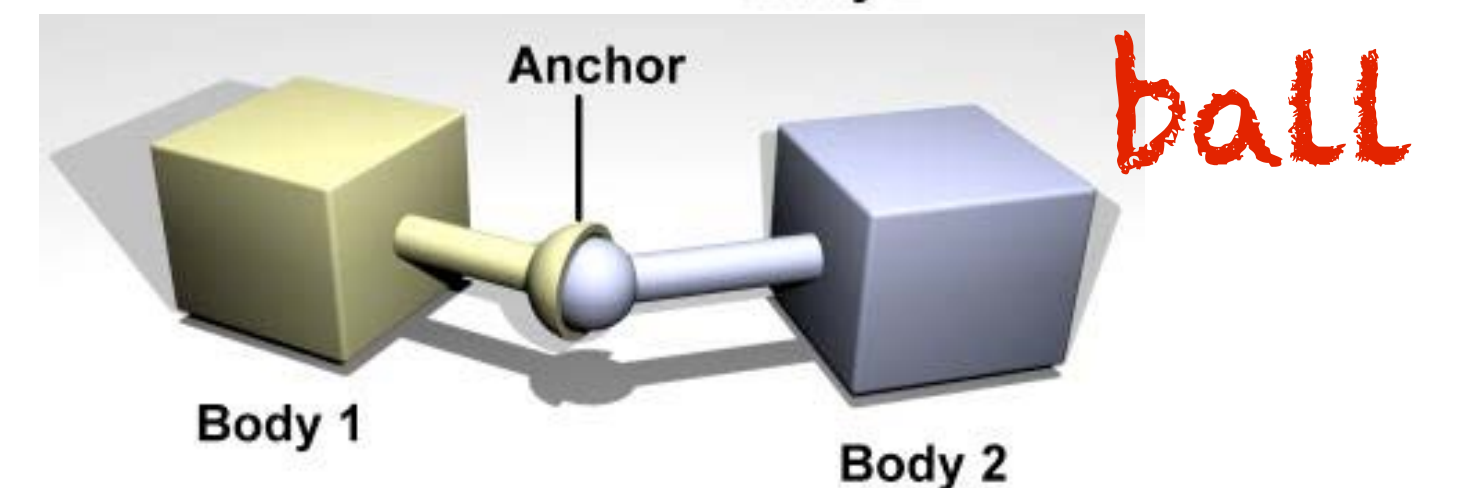
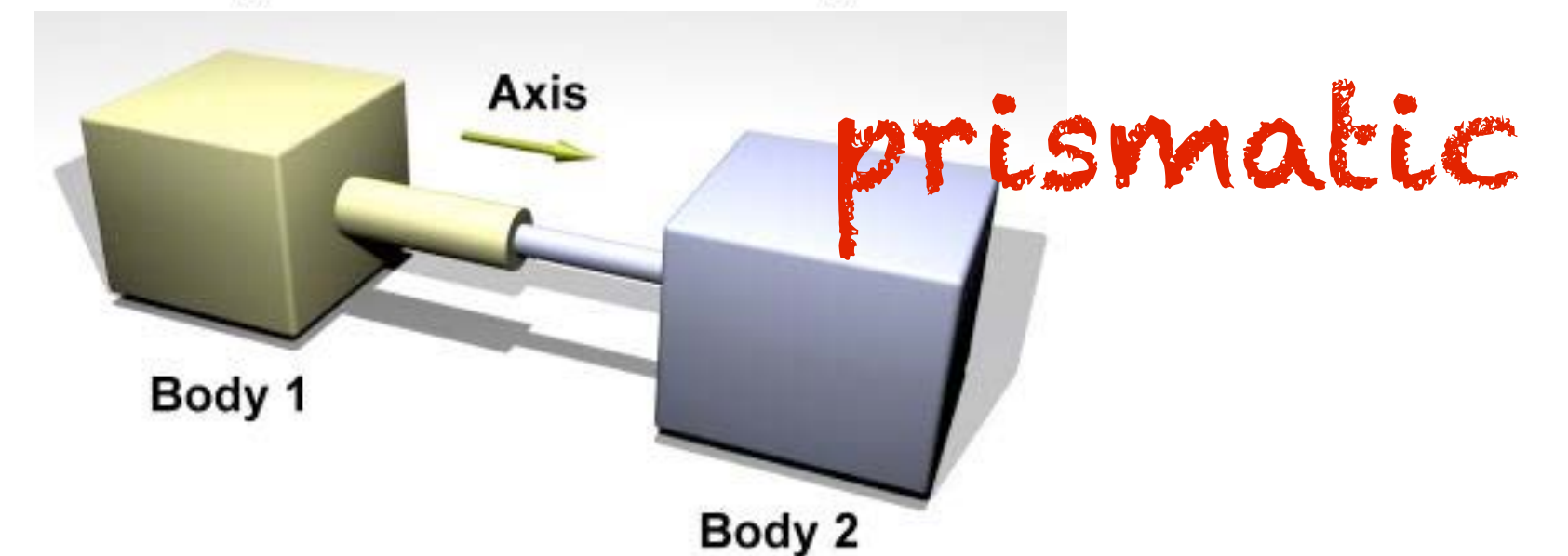
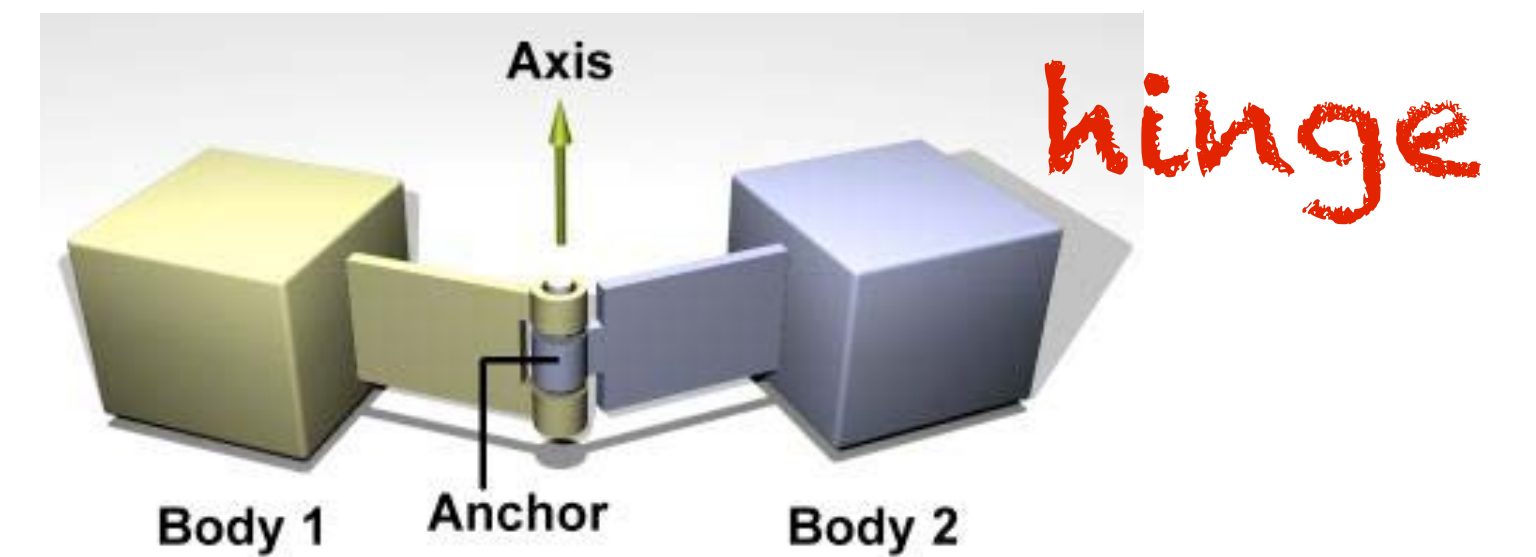
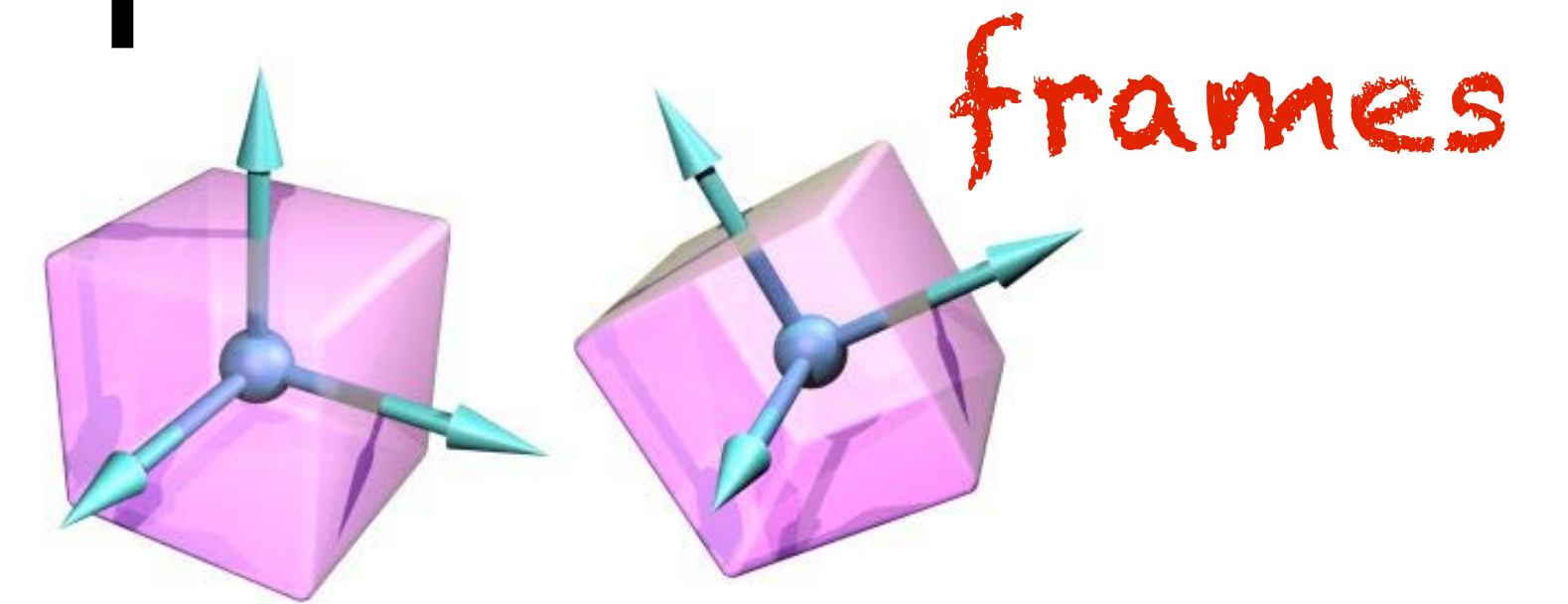


# DOFs and Coordinate Spaces

- Each body has its own frame

**Rigid Body**  
vs.  
**Link**  
vs.  
**Joint**

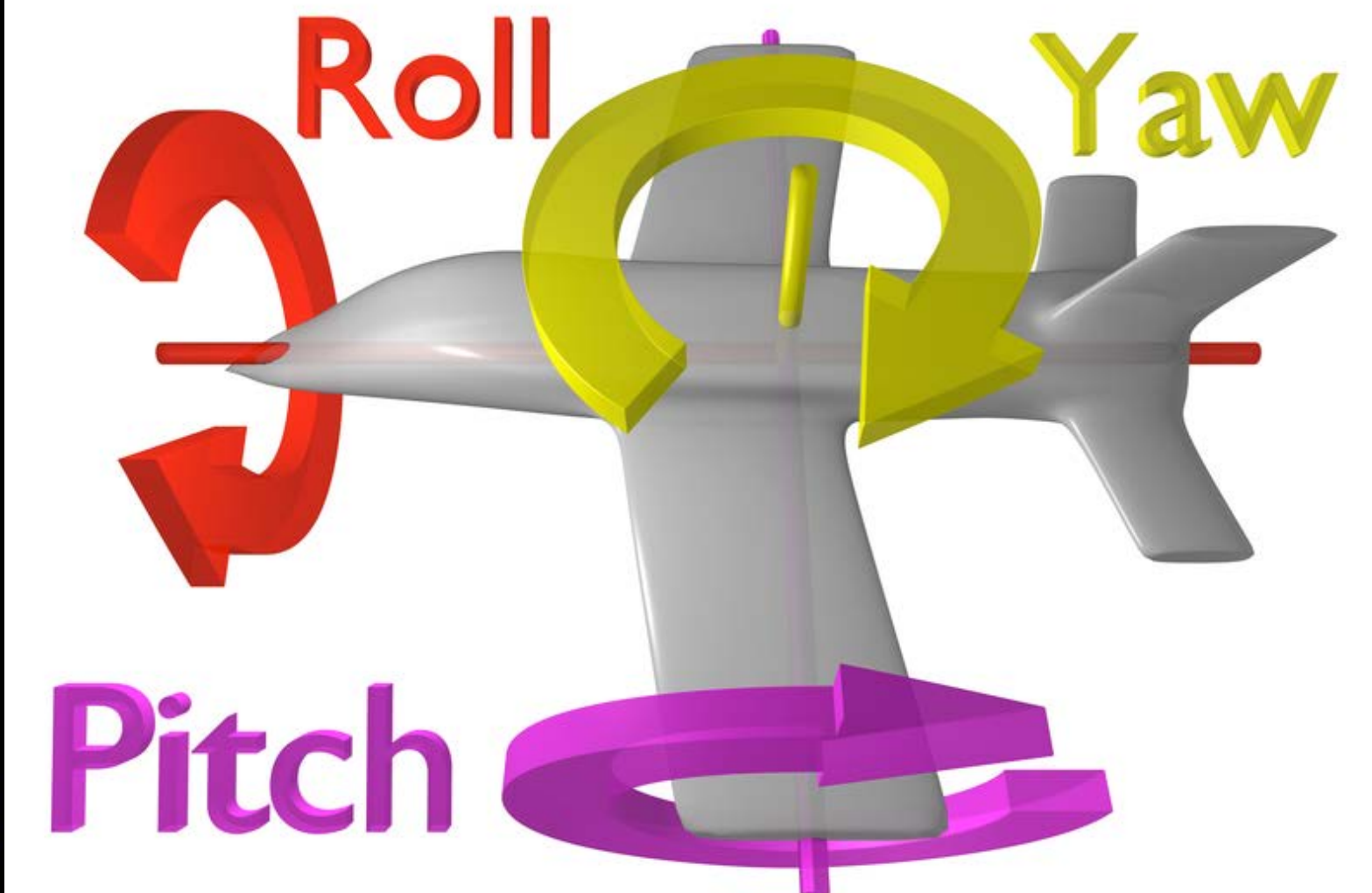
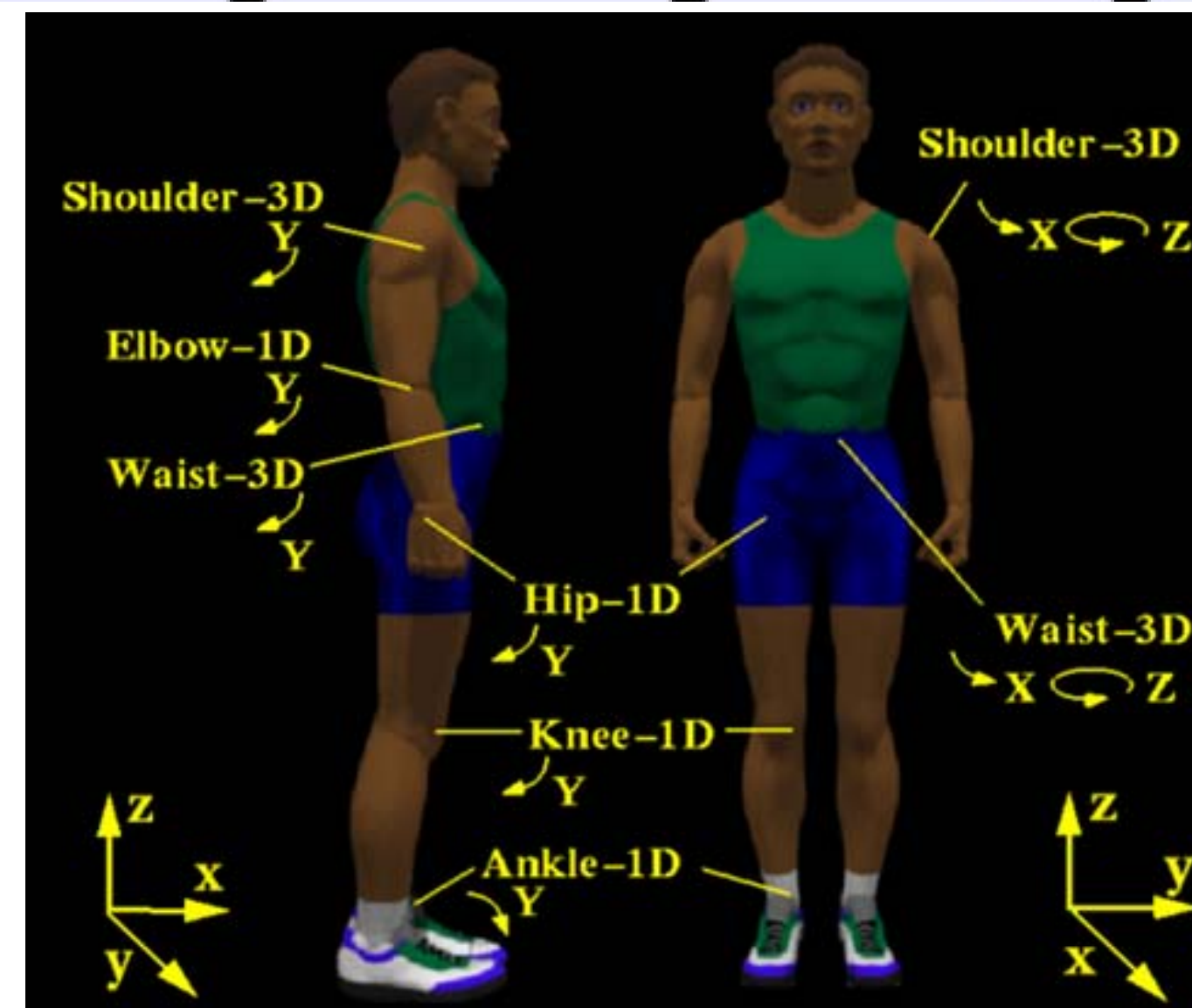
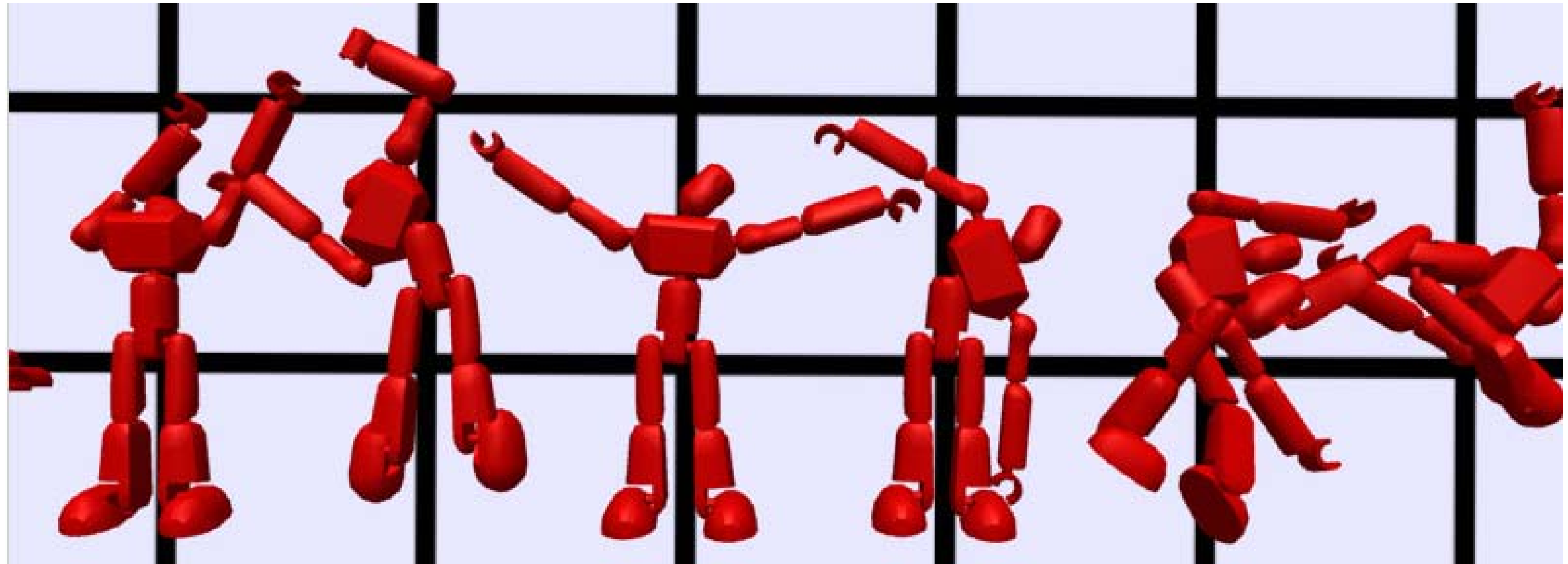
- Spatial geometry attached to each link, but does not affect the body's coordinate frame





# Reset: Kinematics

- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translation and rotation axes of system



# DOF definitions

- Degrees of freedom of a rigid body:  $\sum$  (freedoms of points) – # of independent constraints
- Degrees of freedom of a robot:  $\sum$  (freedoms of bodies) – # of independent constraints

*We can use Grübler's formula to determine the DOF of a robot.  
We will talk about this in the future classes.*



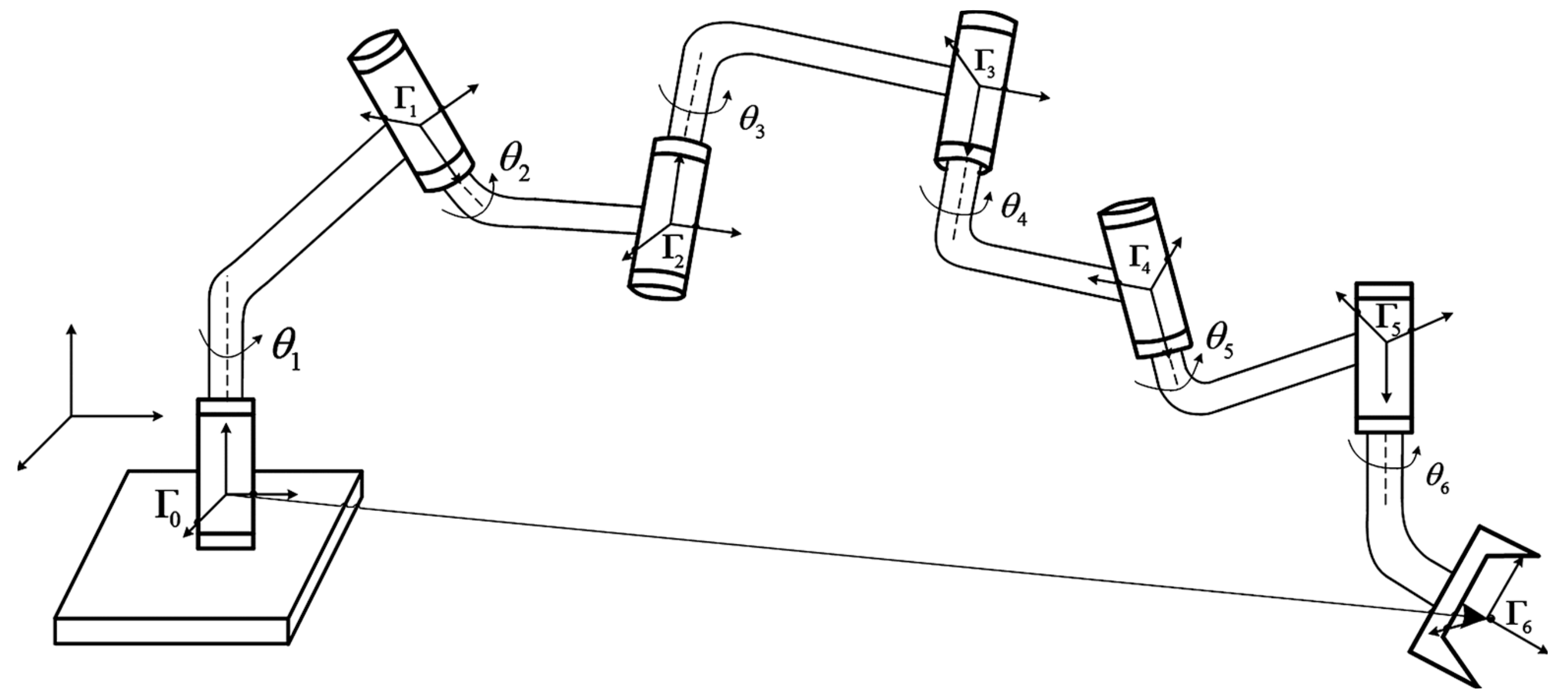


# Robot Kinematics

**Goal:** Given the structure of a robot arm, compute

– **Forward kinematics:** inferring the pose of the end-effector, given the state of each joint.

– **Inverse kinematics:** inferring the joint states necessary to reach a desired end-effector pose.



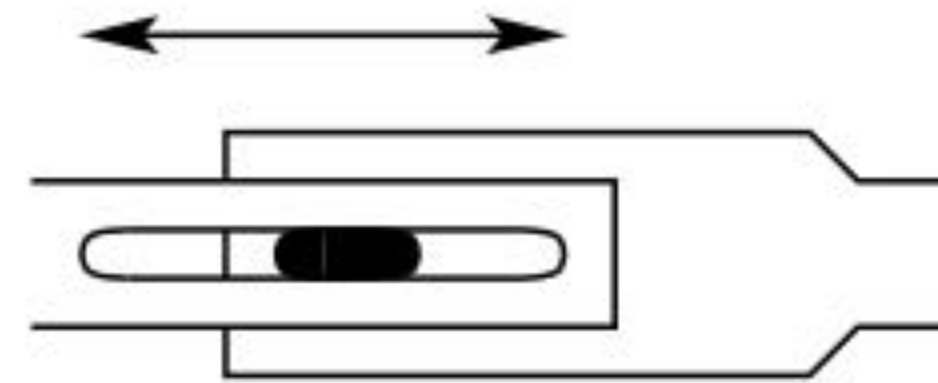
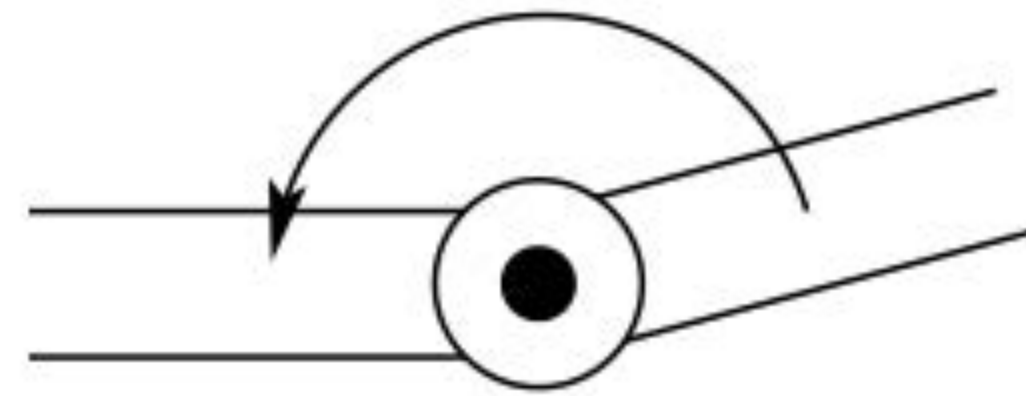
But, we need to start with a linear algebra refresher

# DOF Visual Notation

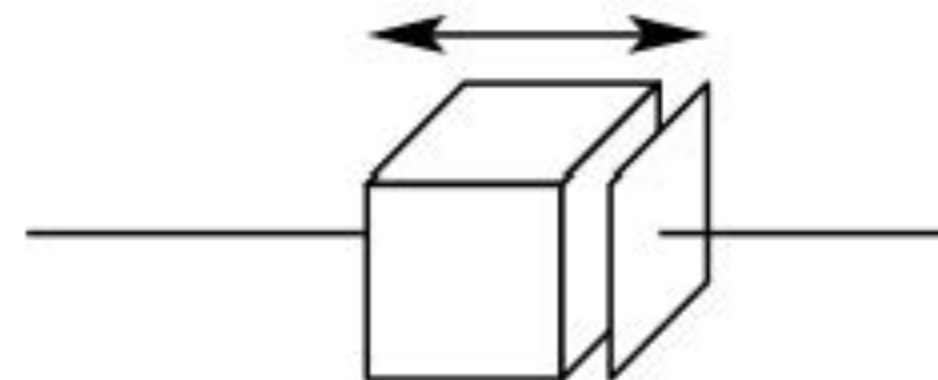
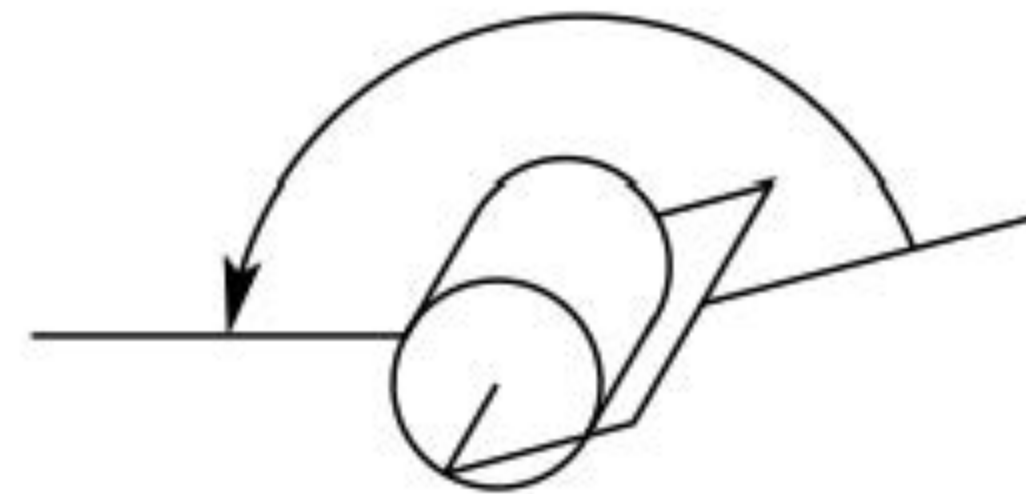
Revolute

Prismatic

2D

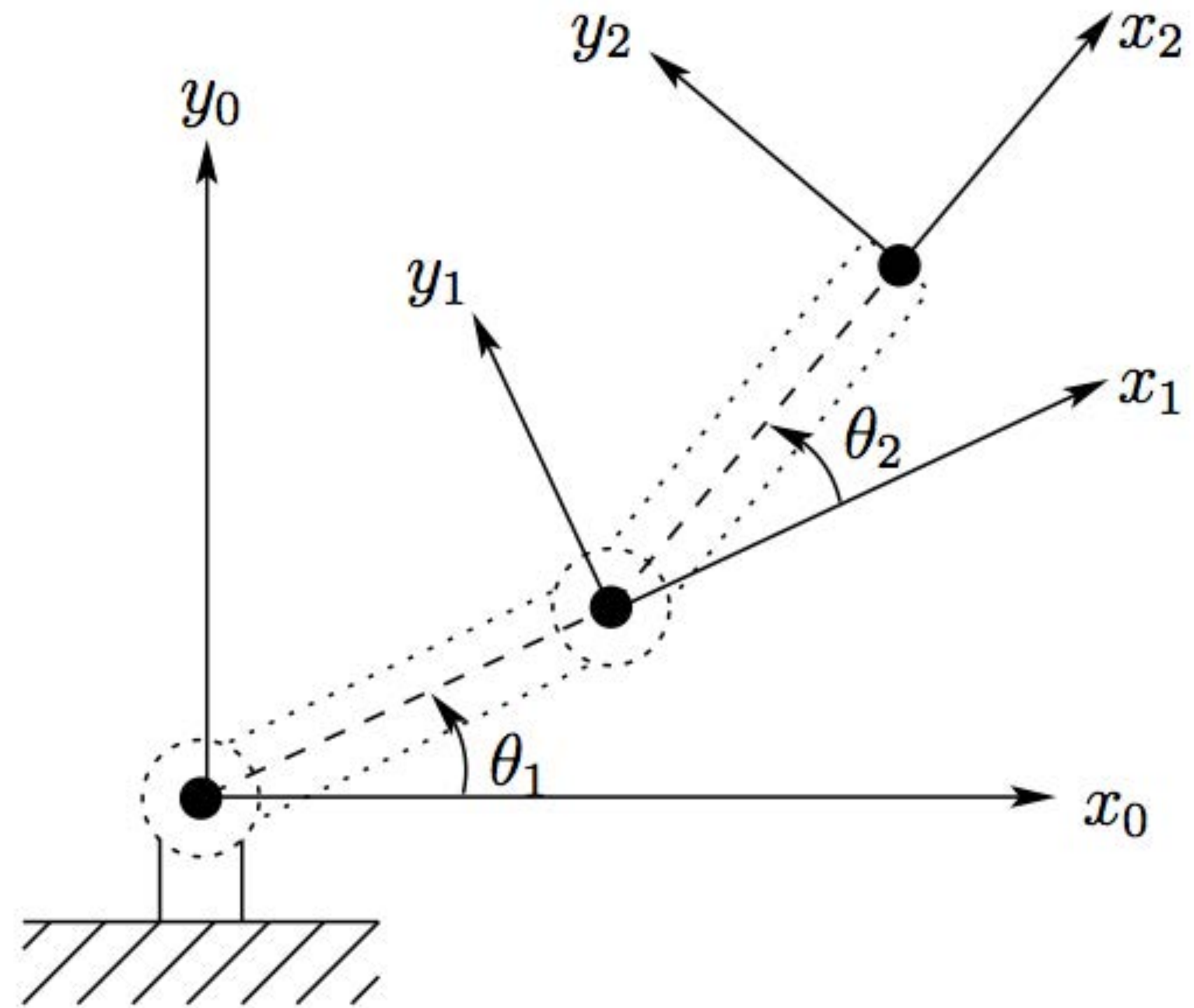
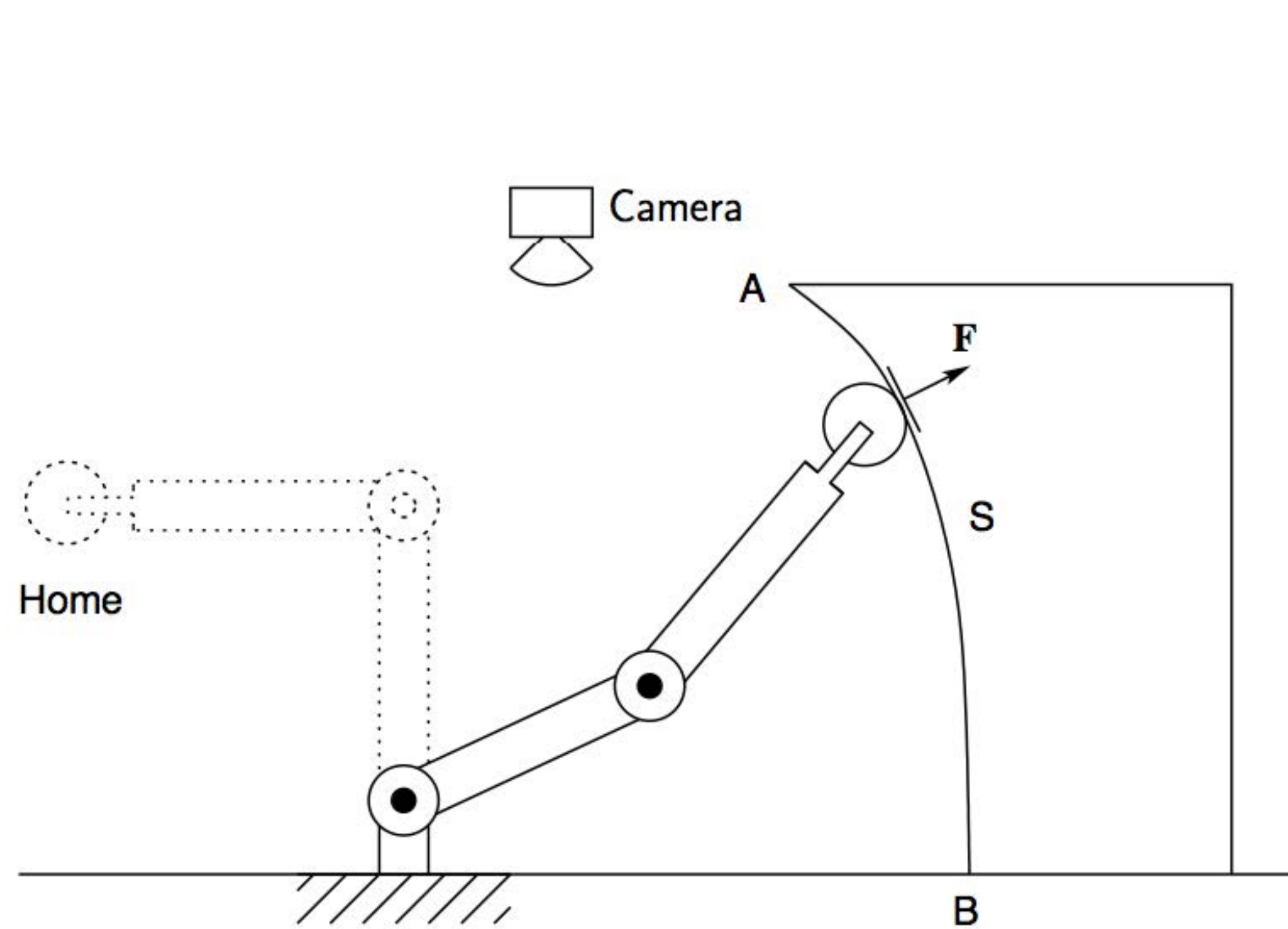


3D



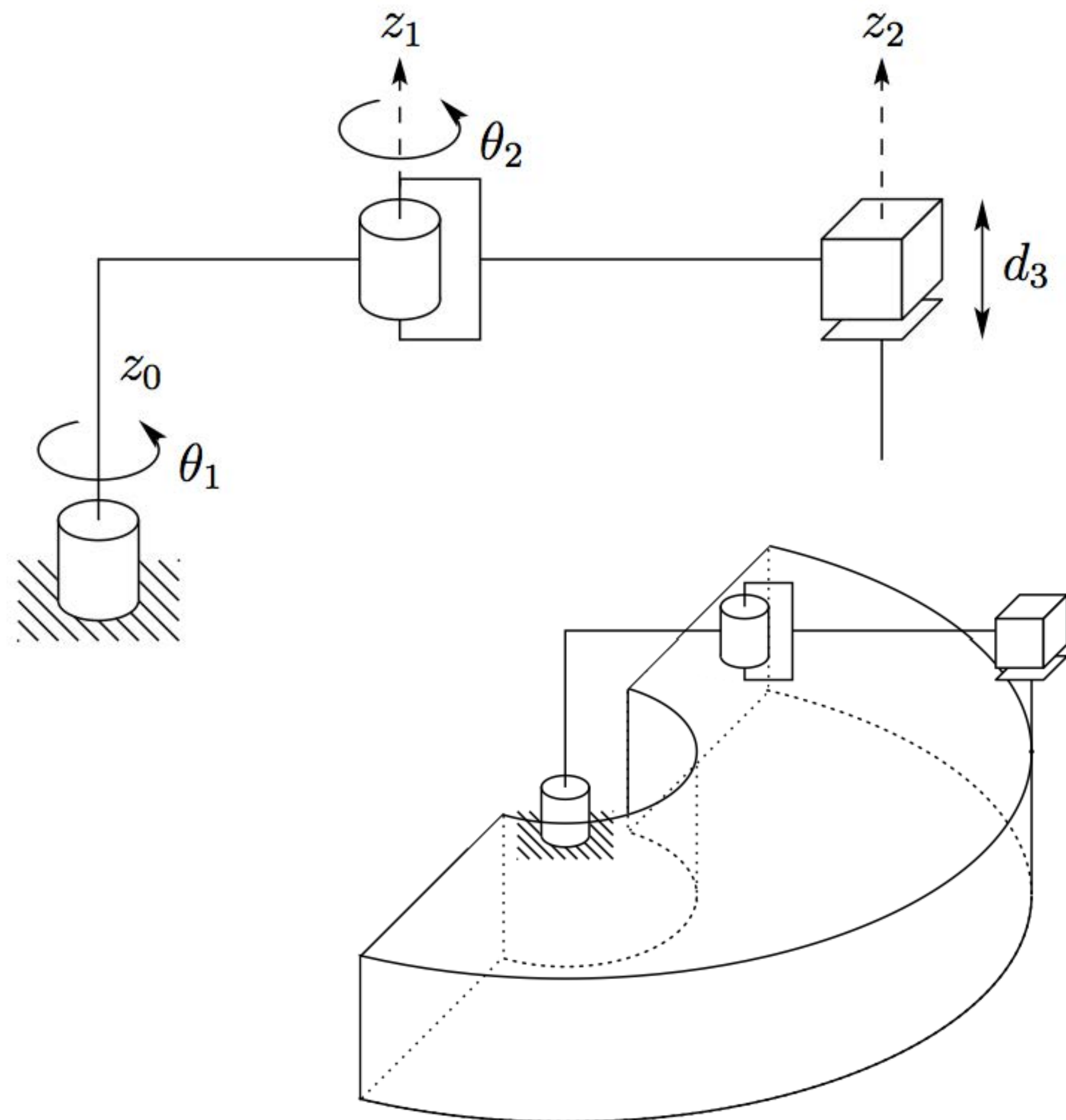


# Planar 3-link Arm (including the ground)



# SCARA Arm

Selective Compliance Assembly Robot Arm

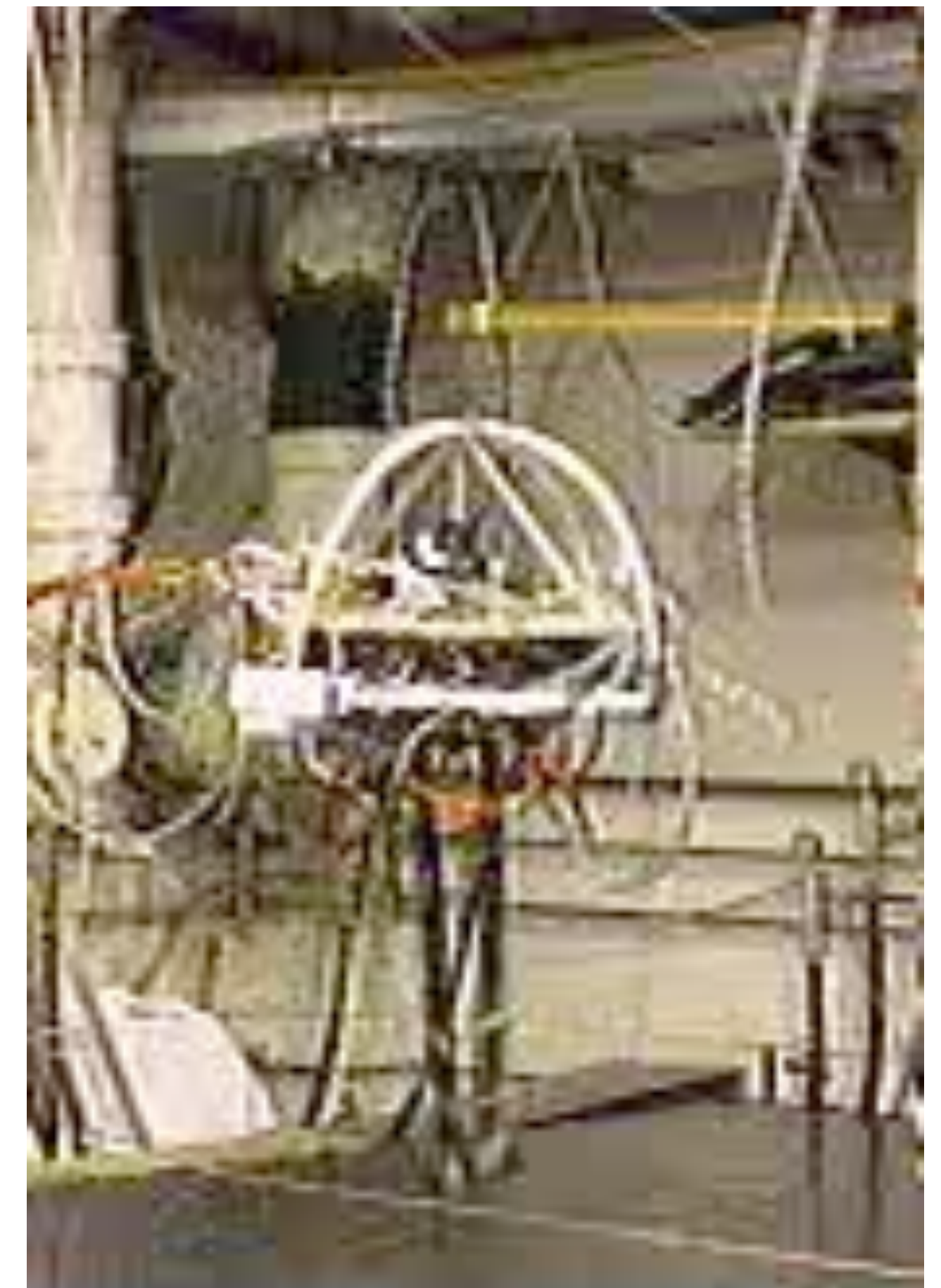
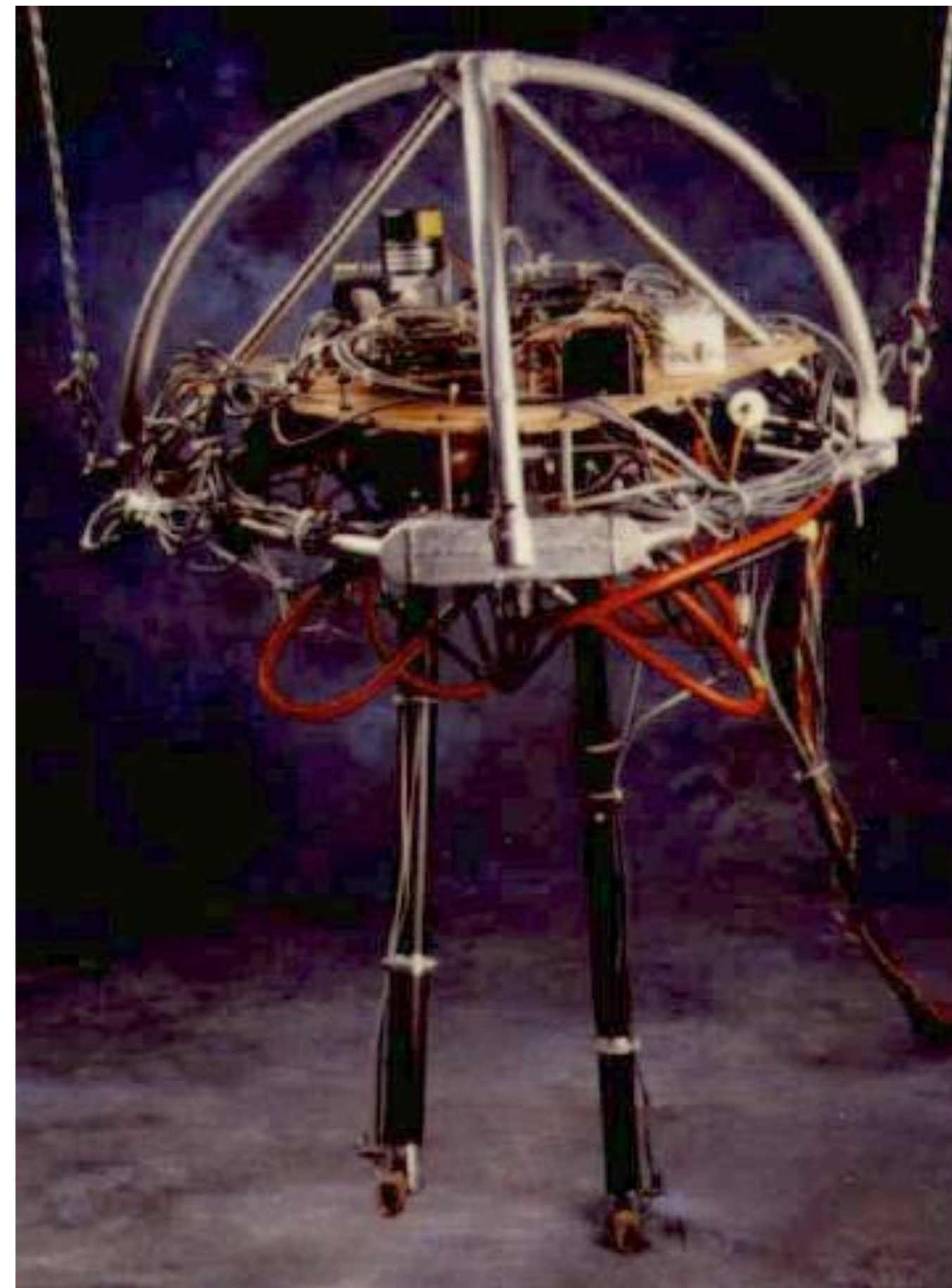
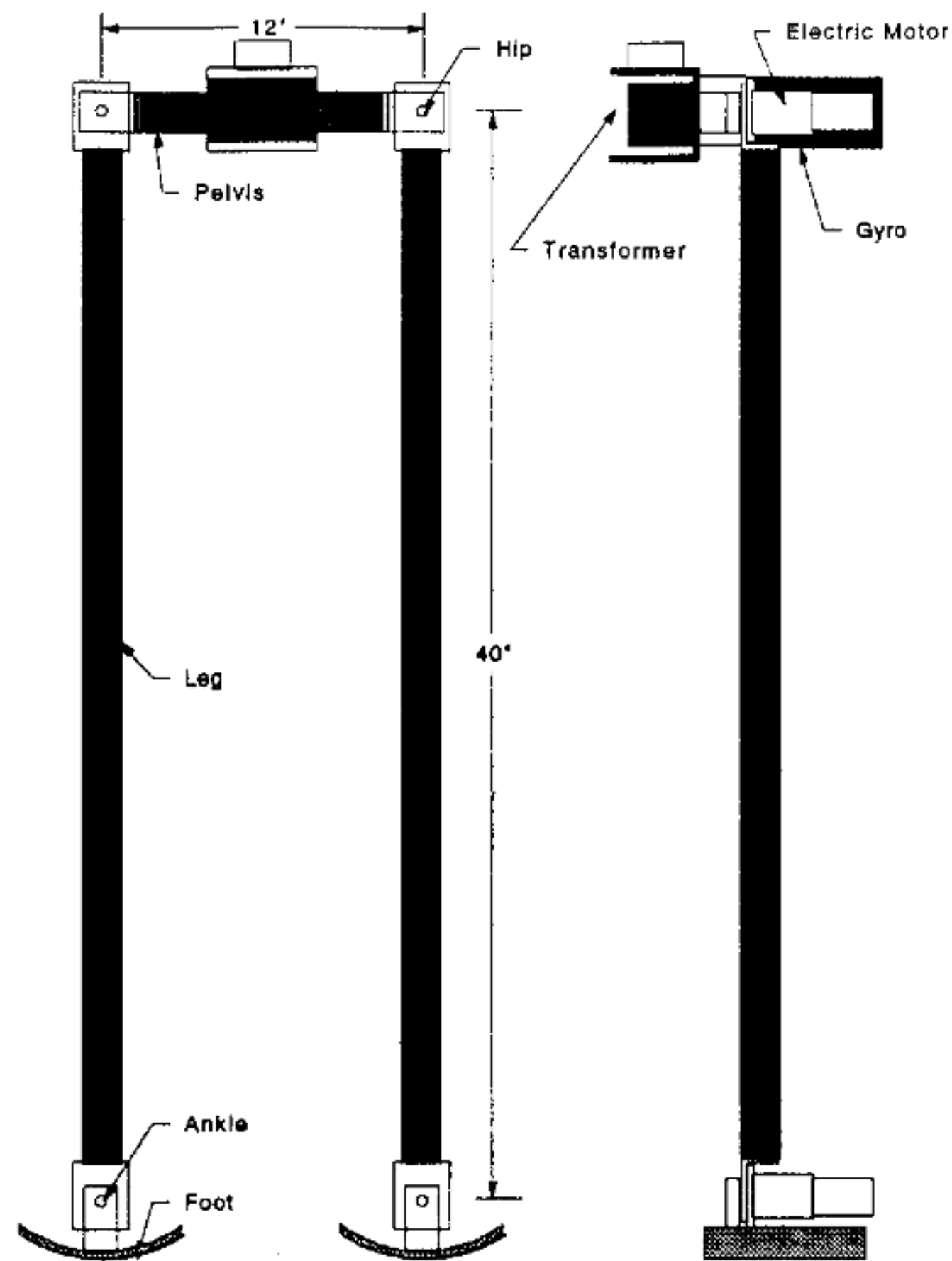


<https://youtu.be/7X5Nmk85kQo>





# Biped Hopper (MIT Leg Lab)



<http://www.ai.mit.edu/projects/leglab/robots/robots.html>



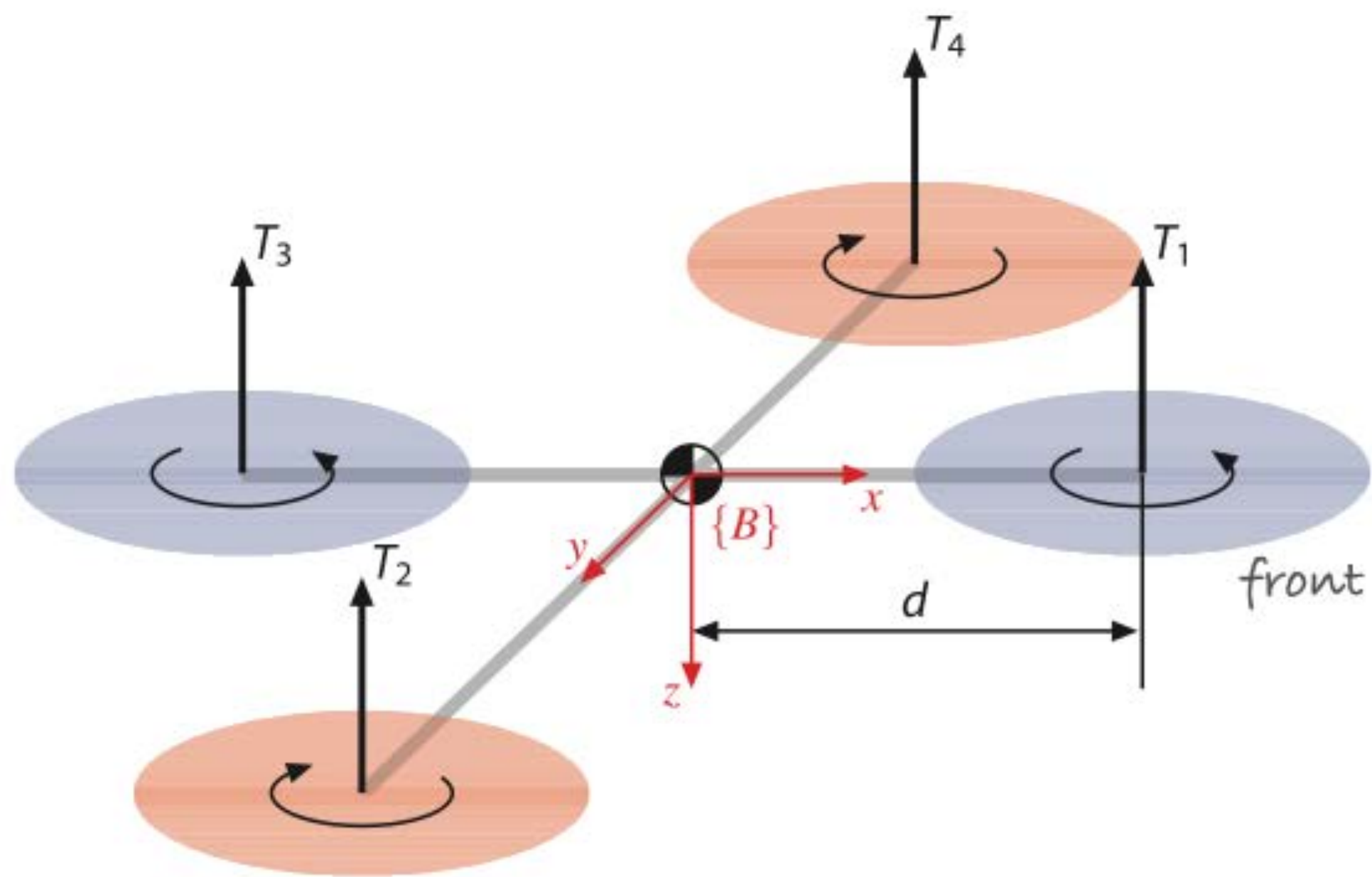


# Big Dog (BDI)





# Quad Rotor Helicopter

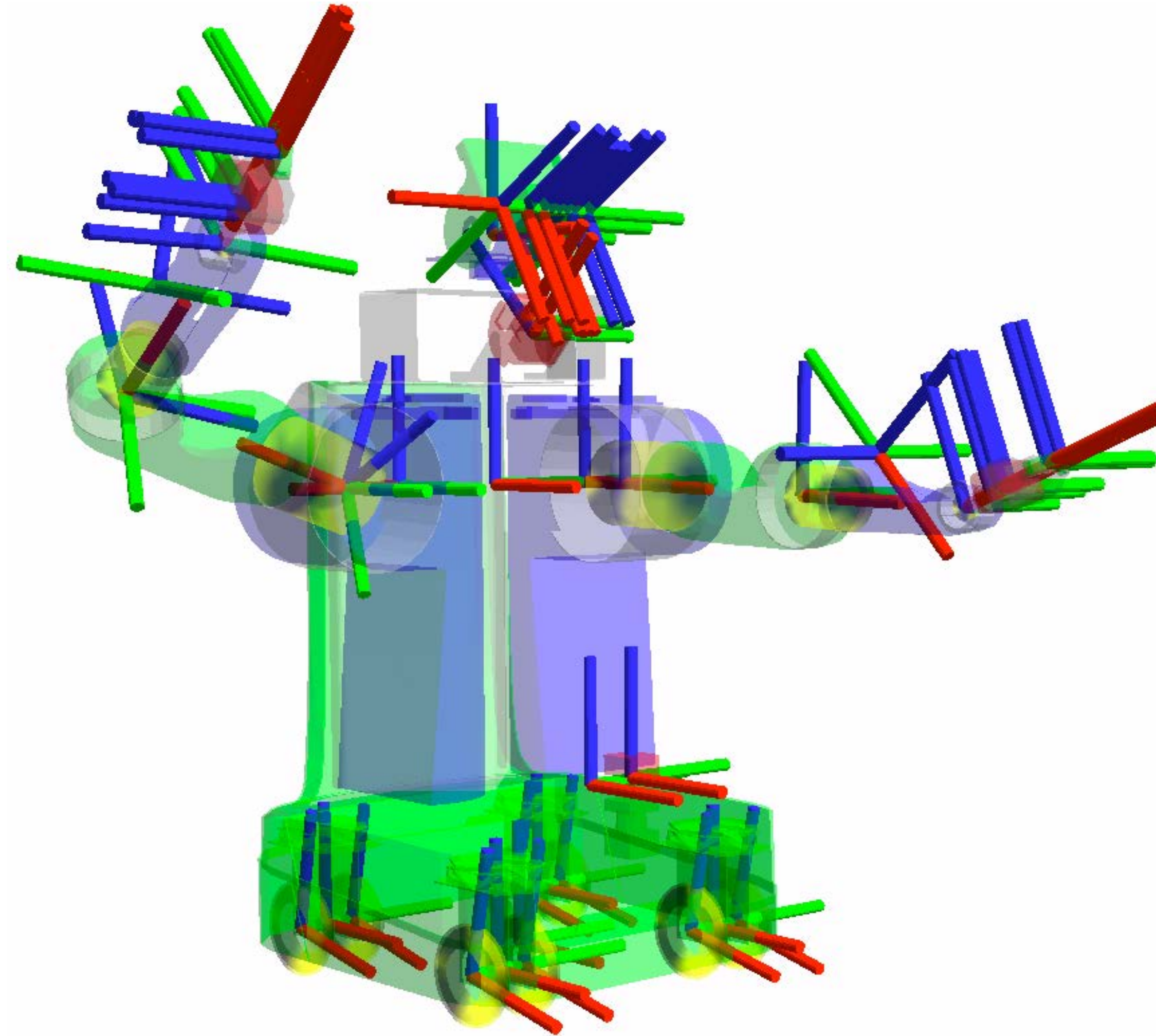


Safety is most important





# PR2

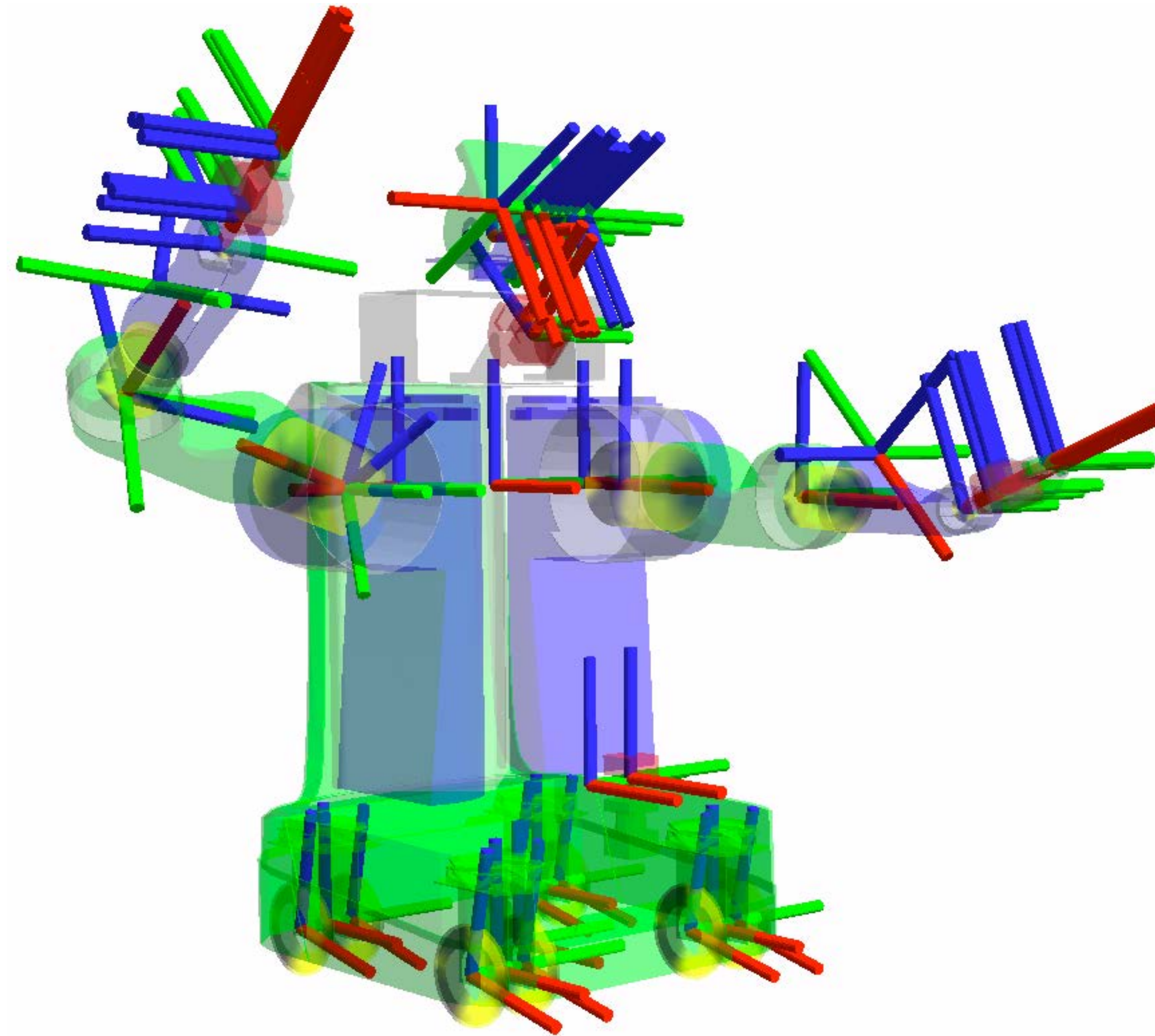


How to express kinematics as the state of an articulated system?





# PR2



How to express kinematics as the state of an articulated system?

We need some math first.



# Algebra

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From Wikipedia, the free encyclopedia

**Algebra** (from [Arabic](#) "*al-jabr*" meaning "reunion of broken parts"<sup>[1]</sup>) is one of the broad parts of [mathematics](#), together with [number theory](#), [geometry](#) and [analysis](#). In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols;<sup>[2]</sup> it is a unifying thread of almost all of mathematics.<sup>[3]</sup> As such, it includes everything from elementary equation solving to the study of abstractions such as [groups](#), [rings](#), and [fields](#). The more basic parts of algebra are

What does algebra provide  
beyond arithmetic?





# Algebra

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From Wikipedia, the free encyclopedia

- Arithmetic applies to addition and multiplication of known numbers
- Algebra includes **abstractions as variables**
  - Unknown numbers or expressions that can take on many values
- An algebra supports addition and multiplication of variables and numbers.
  - For example, from:  $x^2 = 5x - 6$
  - we get:  $(x - 2)(x - 3) = 0$
  - and thus:  $x = 2$  or  $x = 3$ .



# Linear algebra

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From Wikipedia, the free encyclopedia

**Linear algebra** is the branch of [mathematics](#) concerning [vector spaces](#) and [linear mappings](#) between such spaces. Such an investigation is initially motivated by a [system of linear equations](#) containing several unknowns. Such equations are naturally represented using the formalism of [matrices](#) and vectors.<sup>[1]</sup>

What does is linear algebra provide  
beyond algebra?





# Vector space

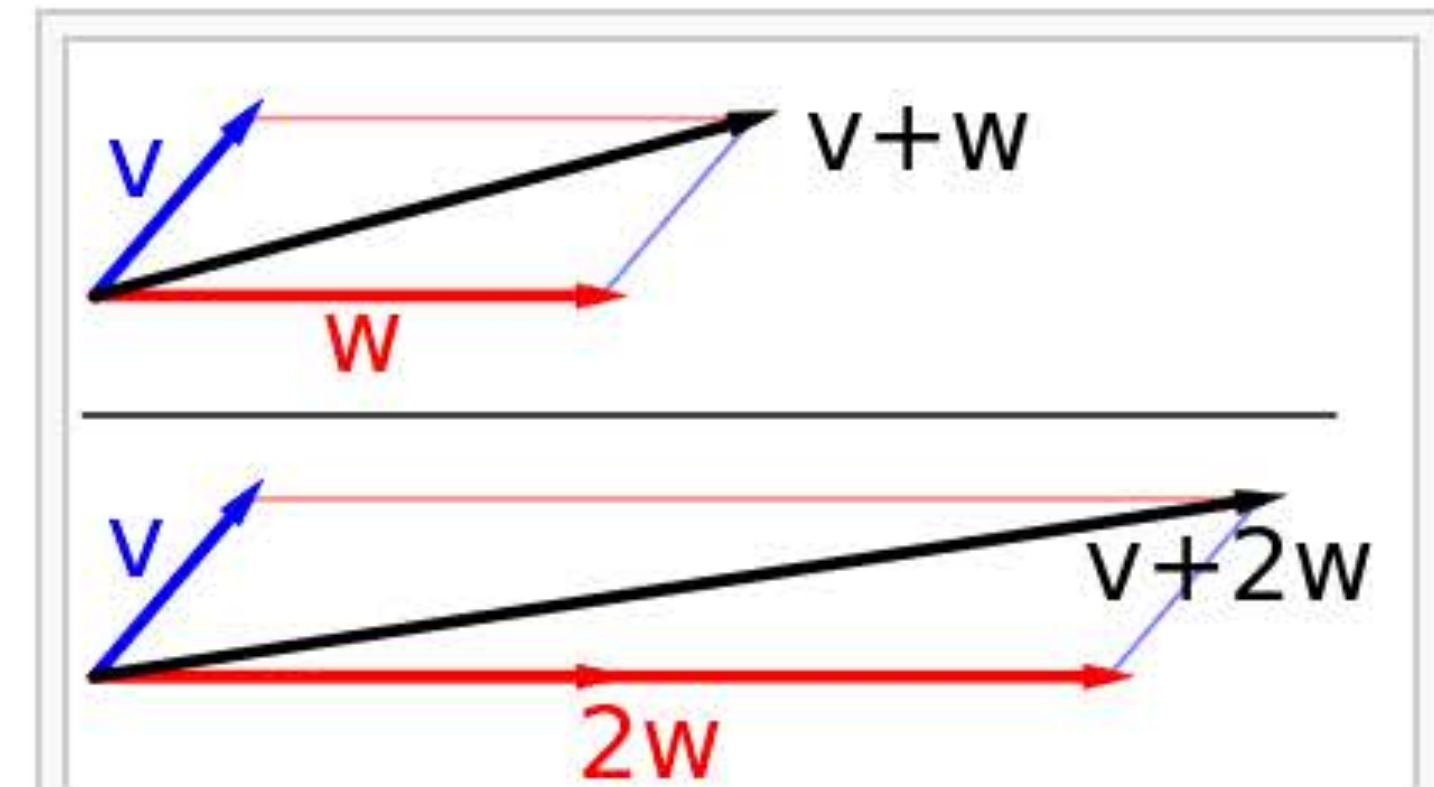
From Wikipedia, the free encyclopedia

*This article is about linear (vector) spaces. For the structure in incidence geometry, see [Linear space \(geometry\)](#).*

A **vector space** (also called a **linear space**) is a collection of objects called **vectors**, which may be **added** together and **multiplied** ("scaled") by numbers, called *scalars* in this context. Scalars are often taken to be **real numbers**, but there are also vector spaces with scalar multiplication by **complex numbers**, **rational numbers**, or generally any **field**. The operations of vector addition and scalar multiplication must satisfy certain requirements, called *axioms*, listed [below](#).

- Describes spaces where vector operations are closed with respect to:

- [REDACTED]
- [REDACTED]



Vector addition and scalar multiplication: a vector  $\mathbf{v}$  (blue) is added to another vector  $\mathbf{w}$  (red, upper illustration). Below,  $\mathbf{w}$  is stretched by a factor of 2, yielding the sum  $\mathbf{v} + 2\mathbf{w}$ .



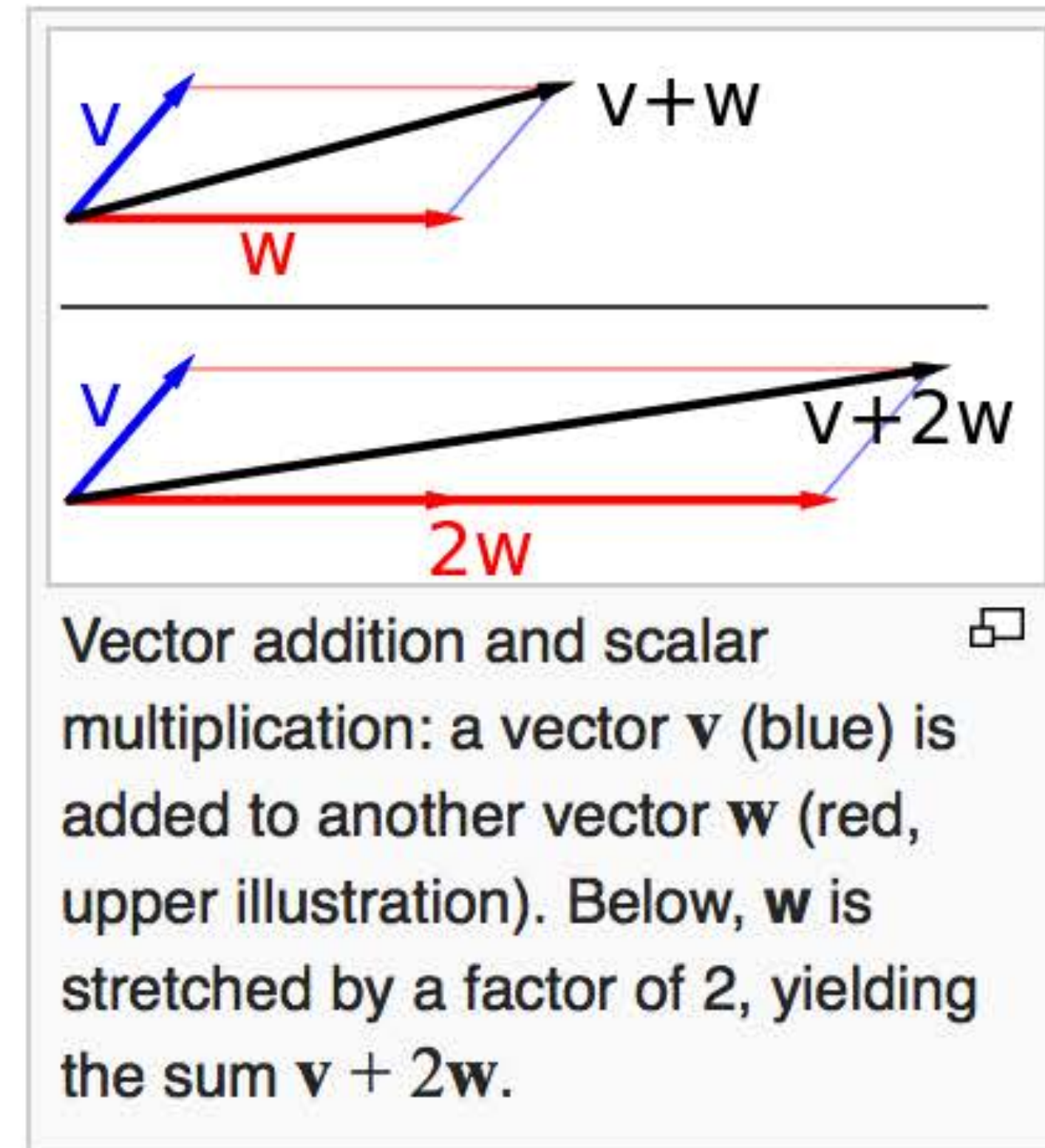
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- Describes spaces where vector operations are closed with respect to:
  - addition
  - scalar multiplication





	Arithmetic	Algebra	Linear Algebra
Abstraction		$x = 3$	$x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$
Addition	$3 + 2 = 5$	$x + 2 = 5$	$x + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$
Scalar multiplication	$3 \times 2 = 6$	$2x = 6$	$2x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$



# Linear algebra

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From Wikipedia, the free encyclopedia

- Many important complex systems are described by collections of linear equations.
- An algebra of scalars, vectors, and matrices helps us work with these systems, keeping track of the complexity.
- Manipulate groups of known and unknown parameters, just like manipulating numbers.
- Linear algebra is essential for representing frames of reference, rotation, translation, and general 3D homogeneous transforms.





# Linear Algebra (Rough) Breakdown

- Geometry of Linear Algebra ← *primary focus for 5551*
  - Vectors, matrices, basic operations, lines, planes, homogeneous coordinates, transformations
- Solving Linear Systems ← *needed for iterative IK*
  - Gaussian Elimination, LU and Cholesky decomposition, over-determined systems, calculus and linear algebra, non-linear least squares, regression
- The Spectral Story
  - Eigensystems, singular value decomposition, principle component analysis, spectral clustering



# Systems of Linear Equations





# Linear algebra

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From Wikipedia, the free encyclopedia

**Linear algebra** is the branch of [mathematics](#) concerning [vector spaces](#) and [linear mappings](#) between such spaces. Such an investigation is initially motivated by a [system of linear equations](#) containing several unknowns. Such equations are naturally represented using the formalism of [matrices](#) and vectors.<sup>[1]</sup>

$$\begin{aligned} 3x + 2y - z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \frac{1}{2}y - z &= 0 \end{aligned}$$

is solved by



# Linear algebra

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# Linear algebra

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linear systems expressed  
in general matrix form

$$A\mathbf{x} = \mathbf{b}$$

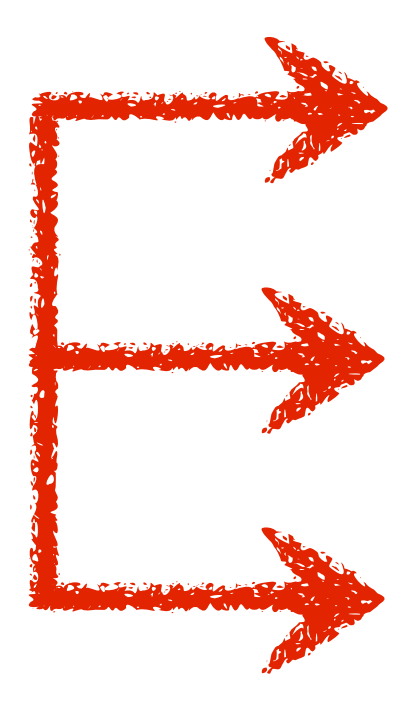
as

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$



M linear equations

each equation yields  
a hyperplane in N-D


$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

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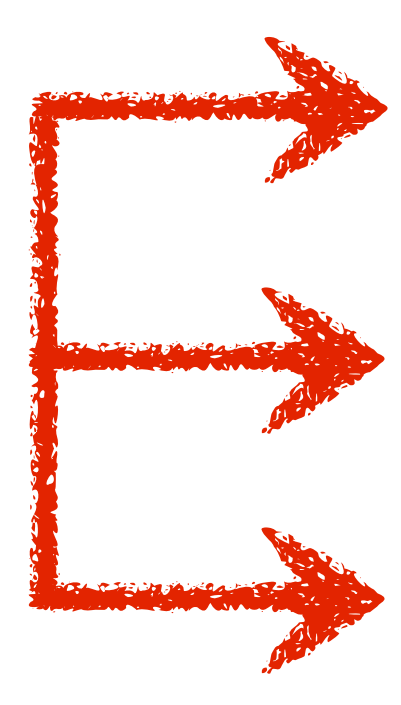


vector of N unknowns to be found



M linear equations

each equation yields  
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$$A \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

If #unknowns > #equations,

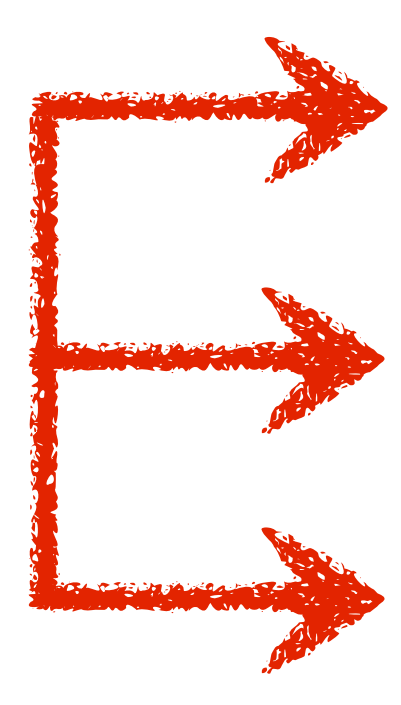
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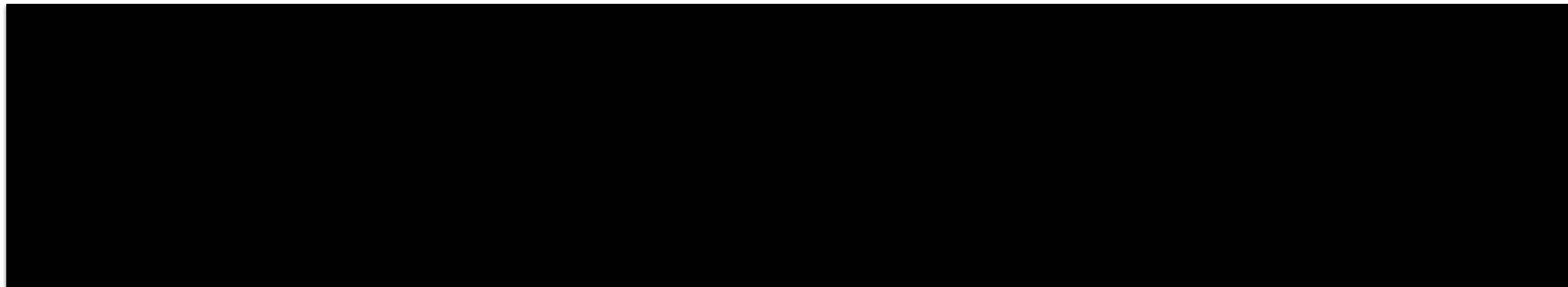


vector of N unknowns to be found

If #unknowns > #equations, underdetermined system, usually with infinite solutions

If #unknowns < #equations,

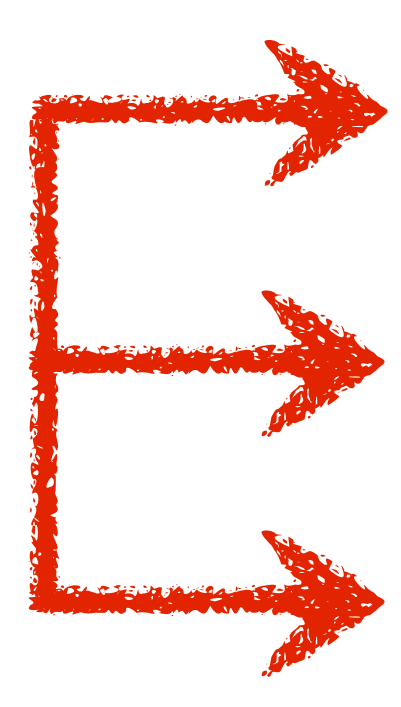
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M linear equations

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$$A \mathbf{x} = \mathbf{b}$$

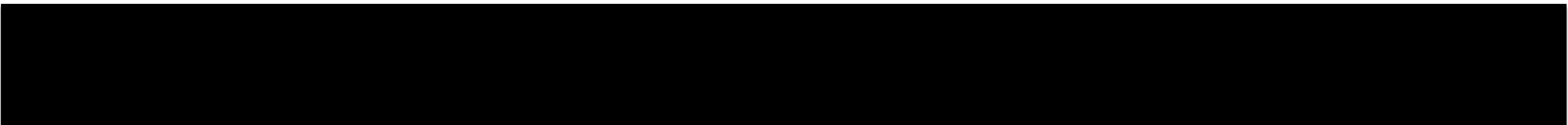


vector of N unknowns to be found

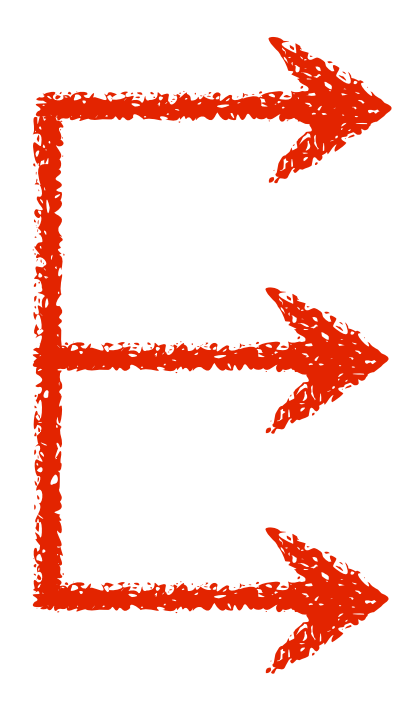
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M linear equations


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each equation yields  
a hyperplane in N-D

$$A \mathbf{x} = \mathbf{b}$$



vector of N unknowns to be found

If #unknowns > #equations, underdetermined system, usually with infinite solutions

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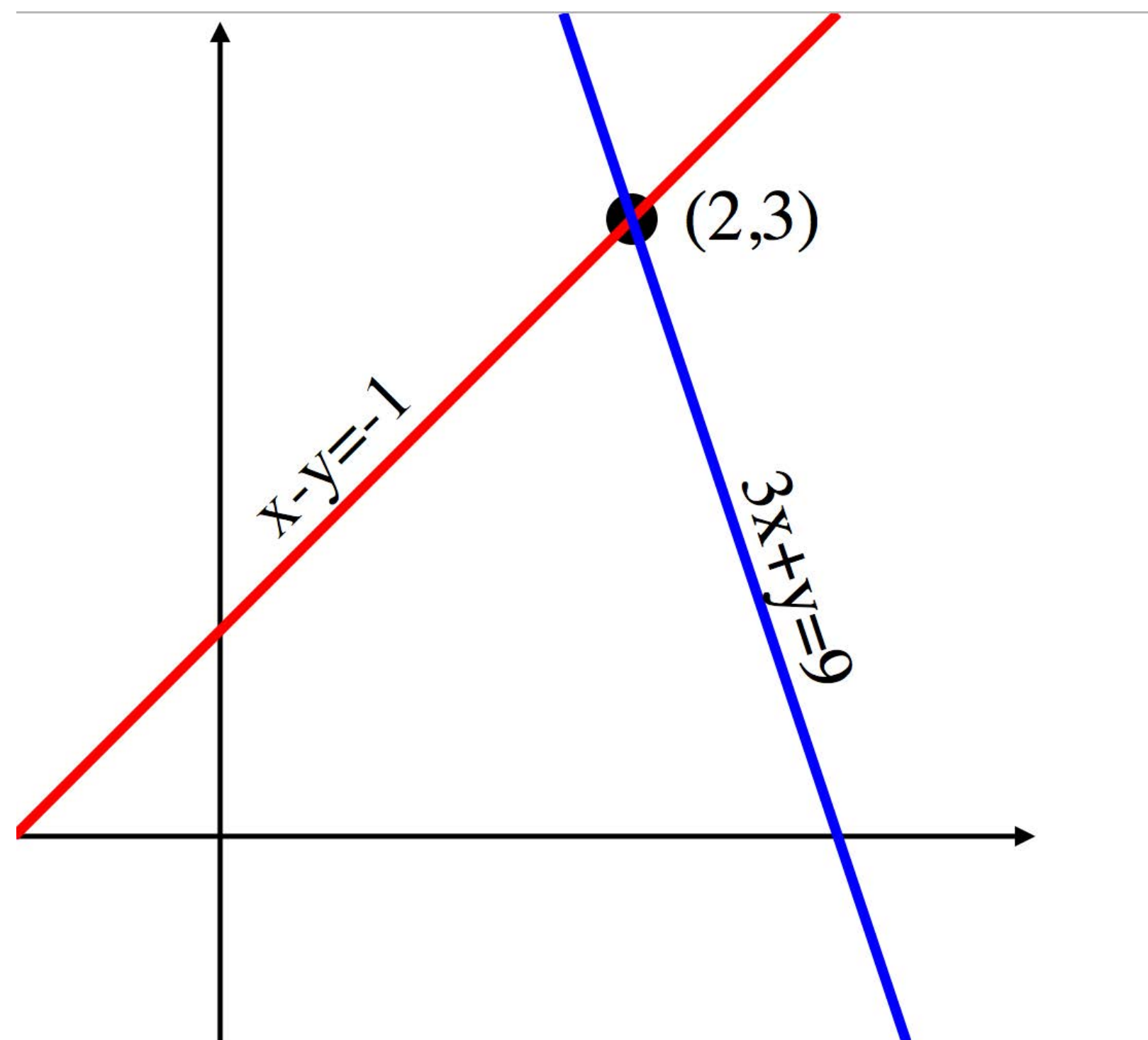
If #unknowns = #equations, usually has a unique solution





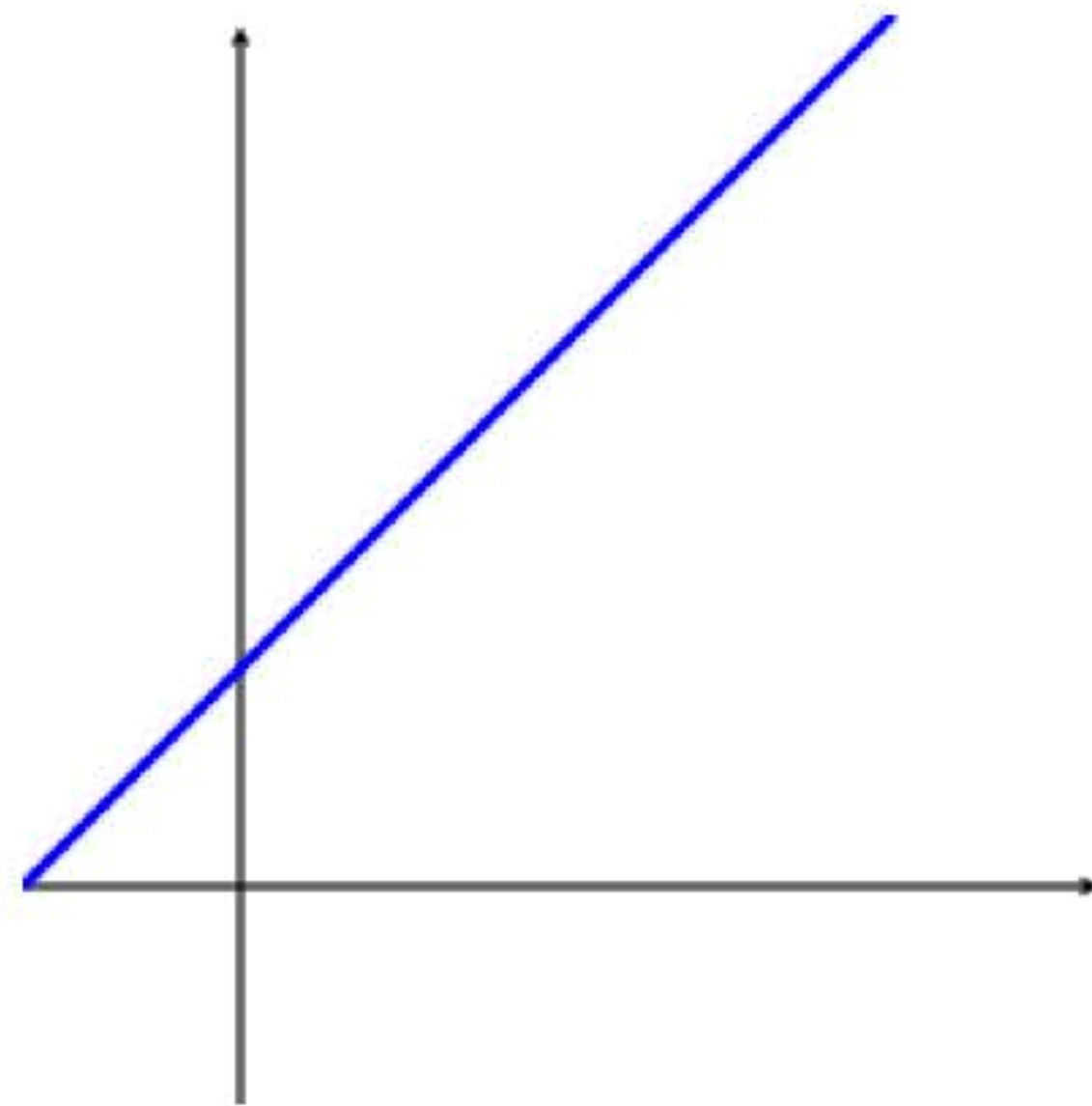
# 2D Example

only single point  
satisfies both lines



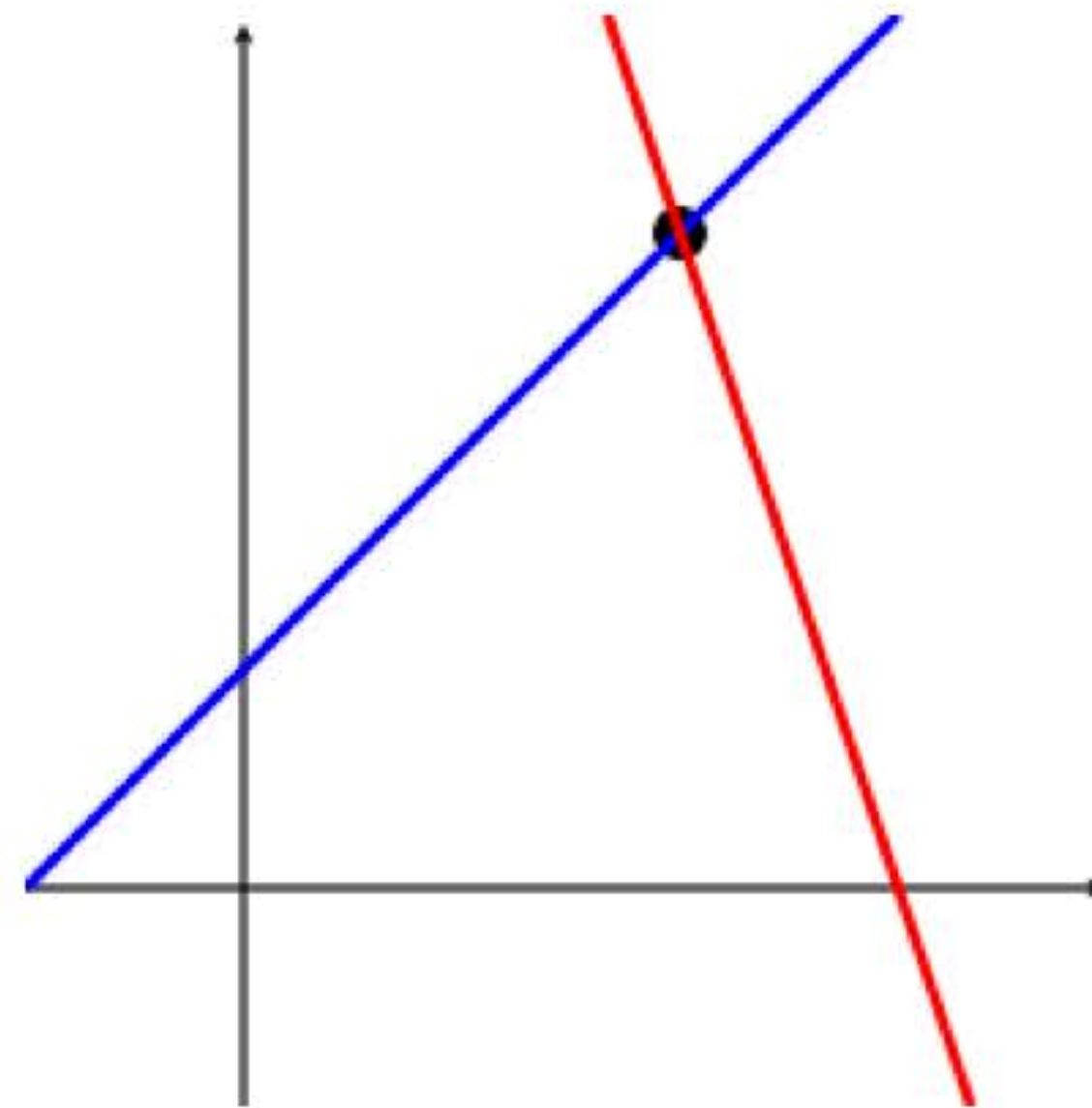
# 2D Example

any point on the line satisfies



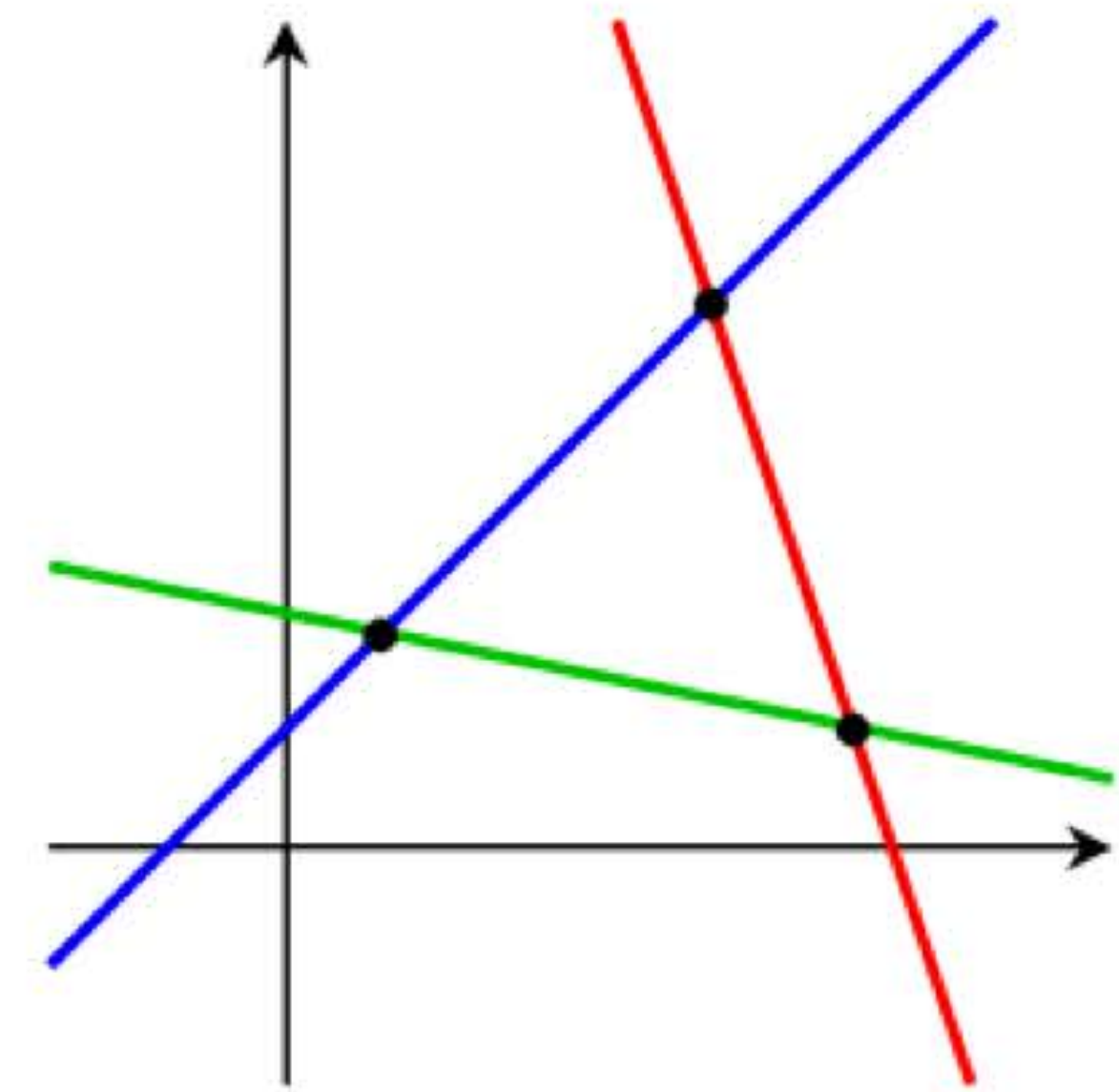
One equation

only single point satisfies both lines



Two equations

no point satisfies all three lines



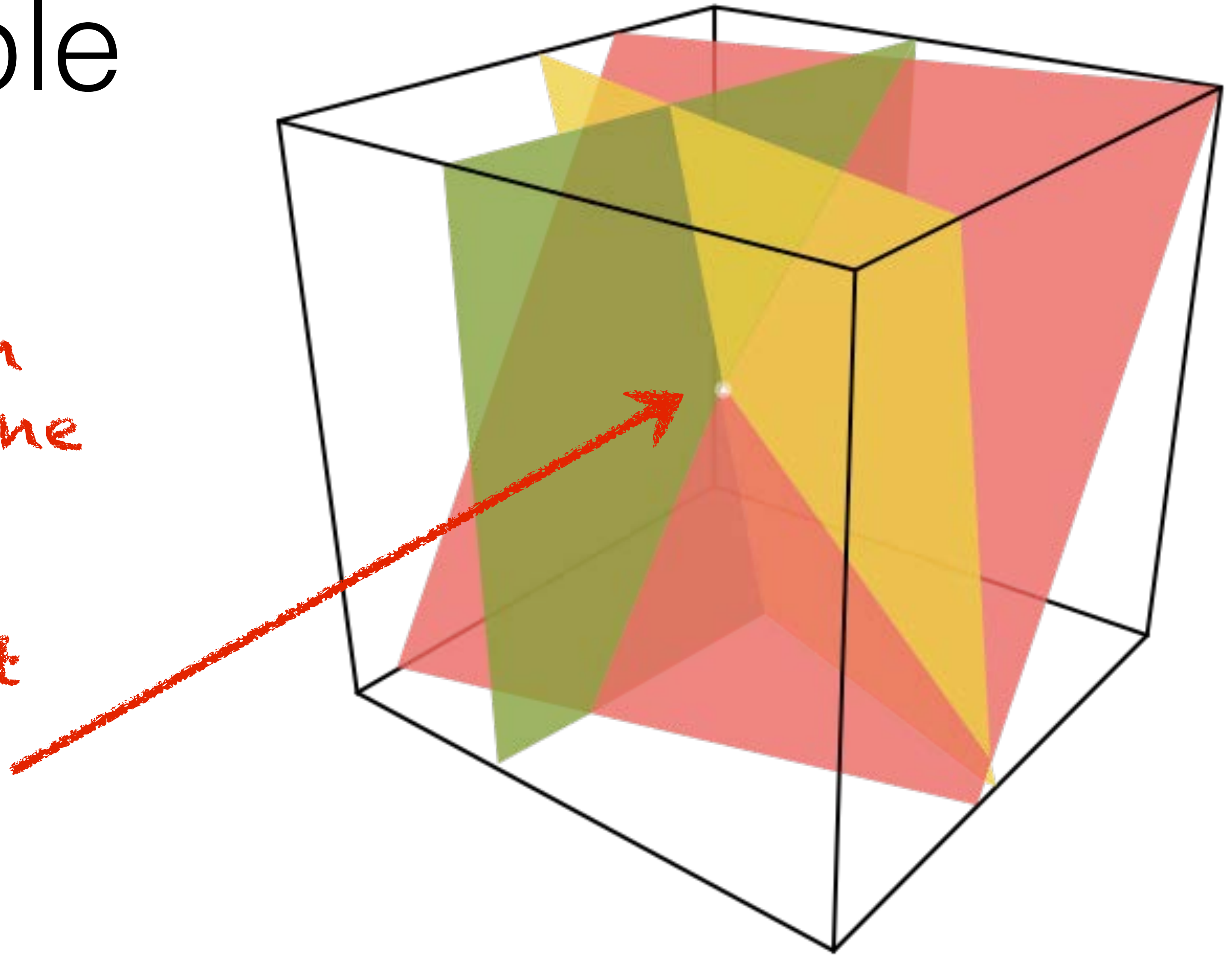
Three equations



# 3D Example

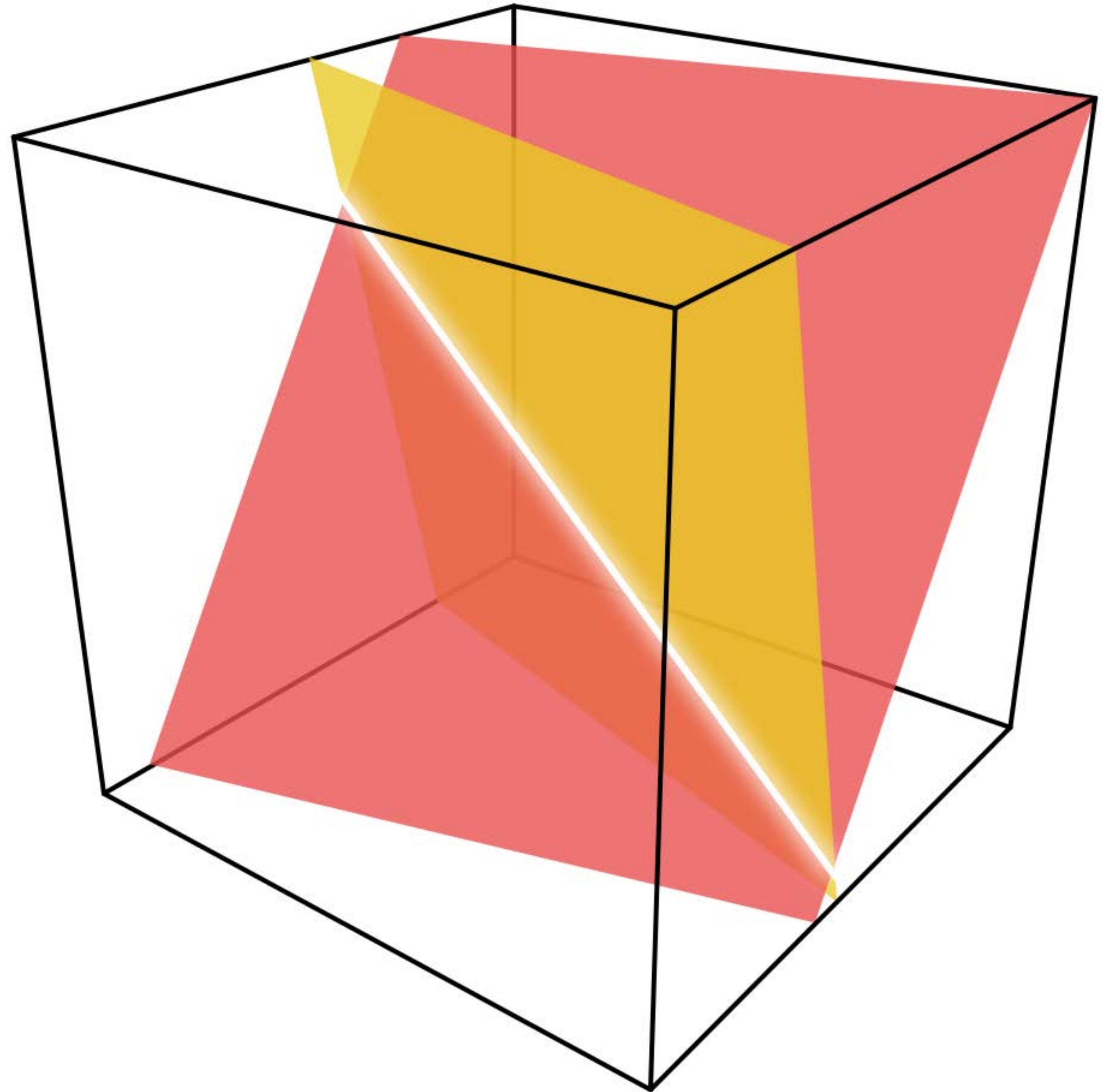
Each equation  
yields a 2D plane  
in 3D space

A single point  
satisfies all  
equations



# 3D Example

How many  
solutions?



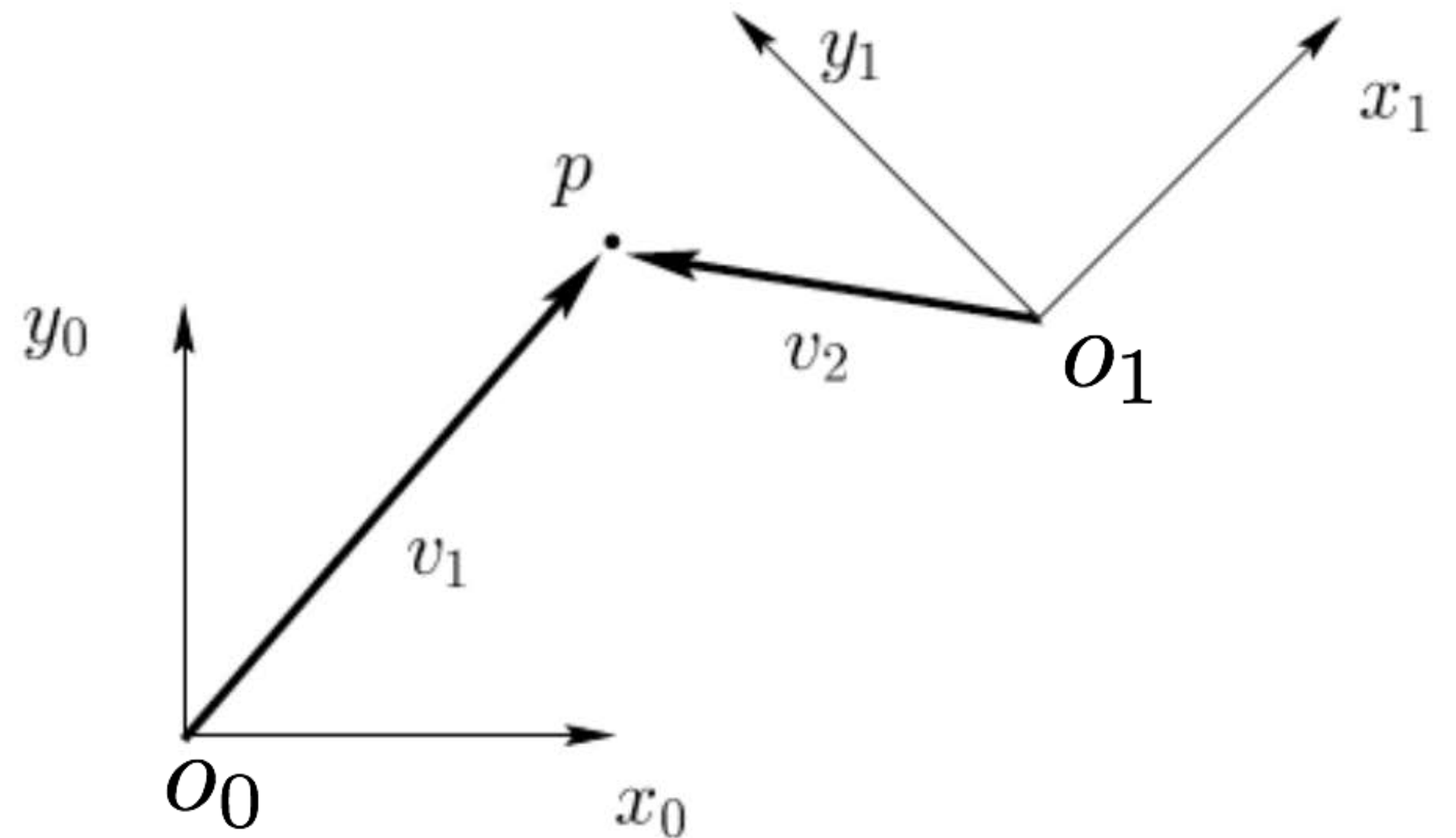


# Coordinate Spaces as Systems of Linear Equations



# Coordinate Spaces (2D)

- Two coordinate frames  $o_0x_0y_0$  and  $o_1x_1y_1$ , and a point  $p$ .
- The location of point  $p$  can be described with respect to either coordinate frame:  $p^0 = [5, 6]^T$  and  $p^1 = [-2.8, 4.2]^T$ .
- The vector  $v_1$  is direction and magnitude from  $o_0$  to  $p$ , and  $v_2$  is from  $o_1$  to  $p$ .

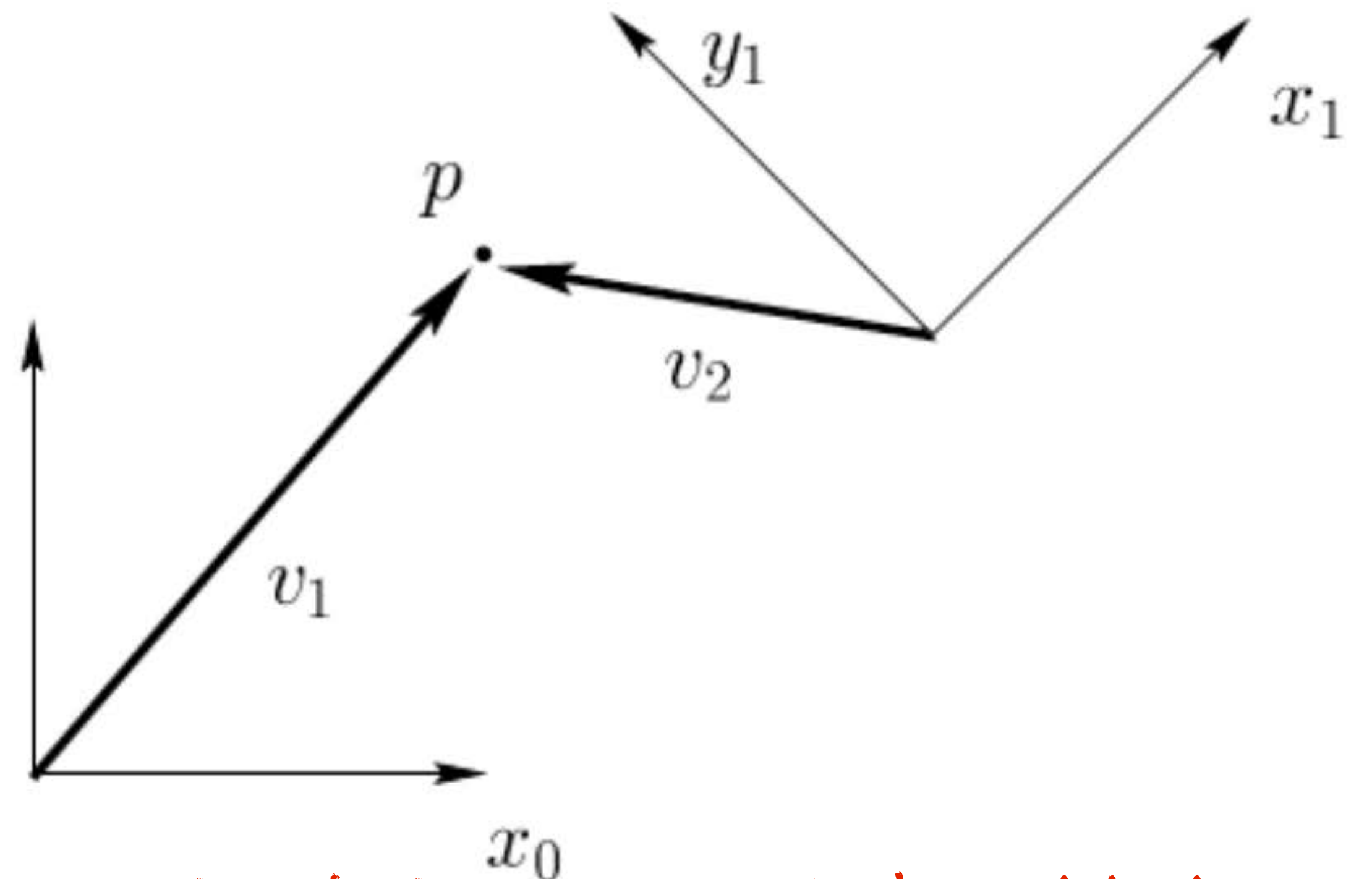




# Coordinate Spaces (2D)

- Point  $p$  has a location.
- Vectors  $v_1$  and  $v_2$  have directions and magnitudes.

- $v_1^0 = [5, 6]^T$  vector 1 in frame 0
- $v_1^1 = [7.77, 0.8]^T$  vector 1 in frame 1
- $v_2^0 = [-5.1, 1]^T$  vector 2 in frame 0
- $v_2^1 = [-2.8, 4.2]^T$  vector 2 in frame 1

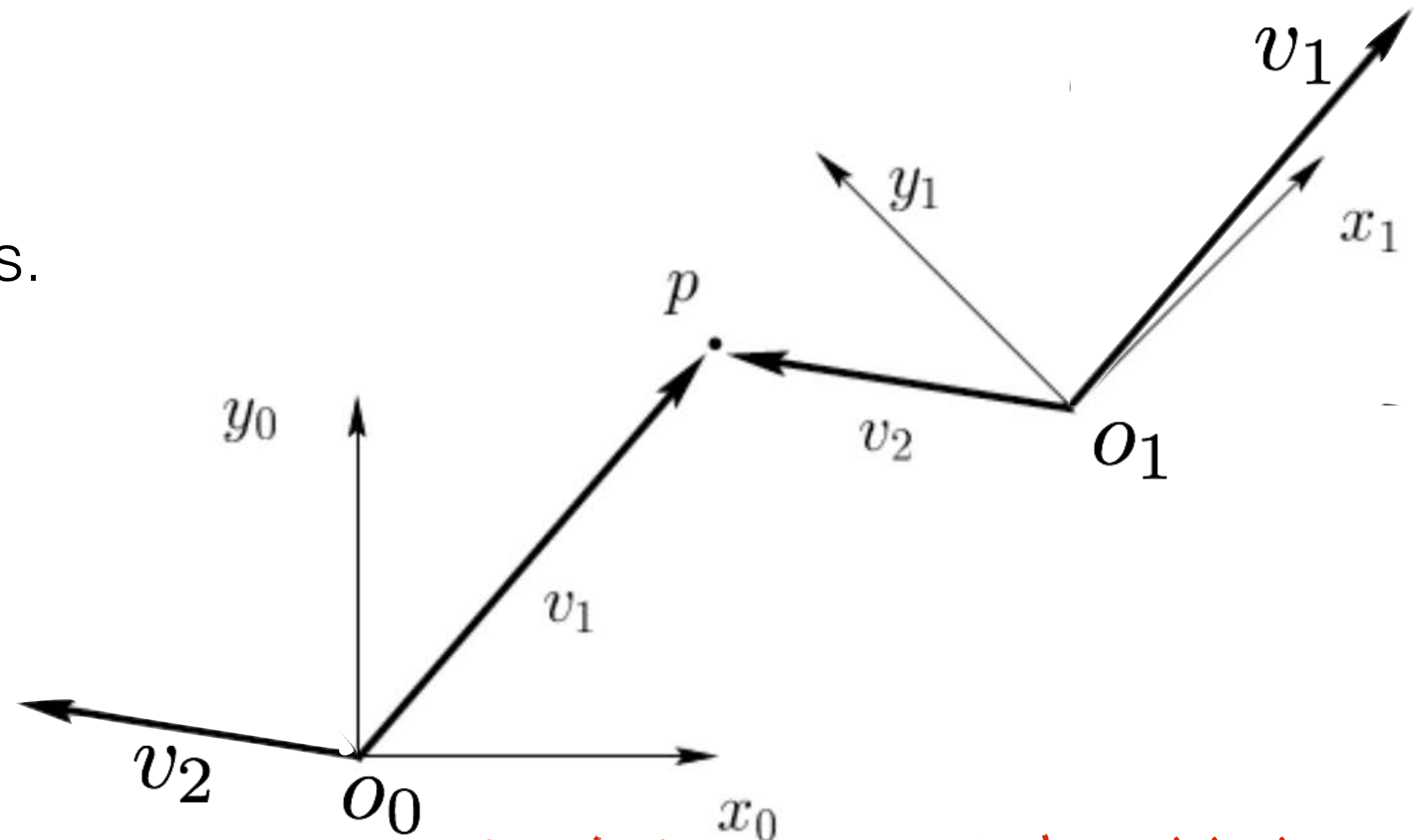


Note: Vectors can only be added when they are in the same coordinate frame.

# Coordinate Spaces (2D)

- Point  $p$  has a location.
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- $v_2^0 = [-5.1, 1]^T$
- $v_2^1 = [-2.8, 4.2]^T$



Note: Vectors can only be added when they are in the same coordinate frame.



# Vectors and Matrices

N-dimensional vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

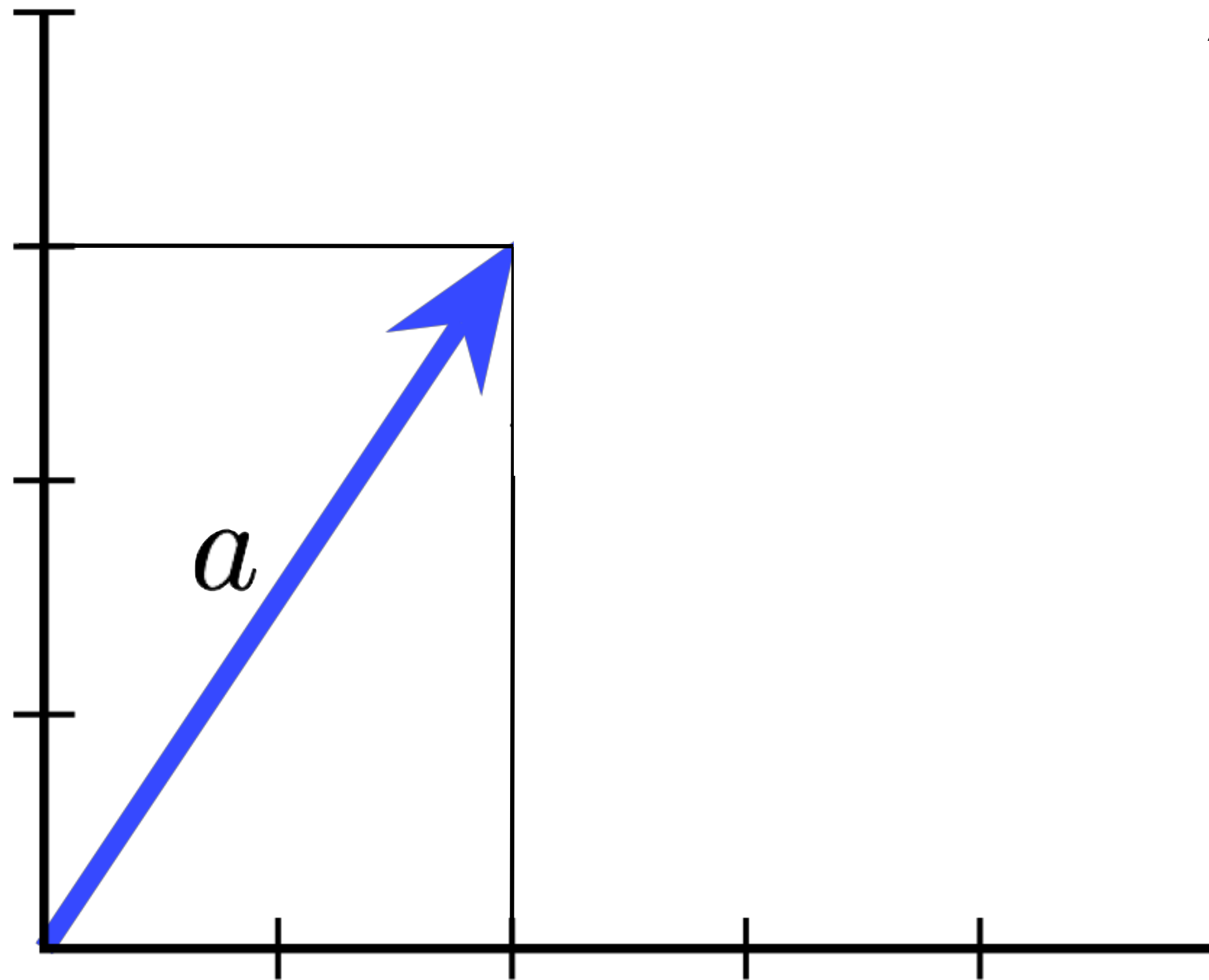
M-by-N matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$



# 2D Vector

A vector is a motion in space

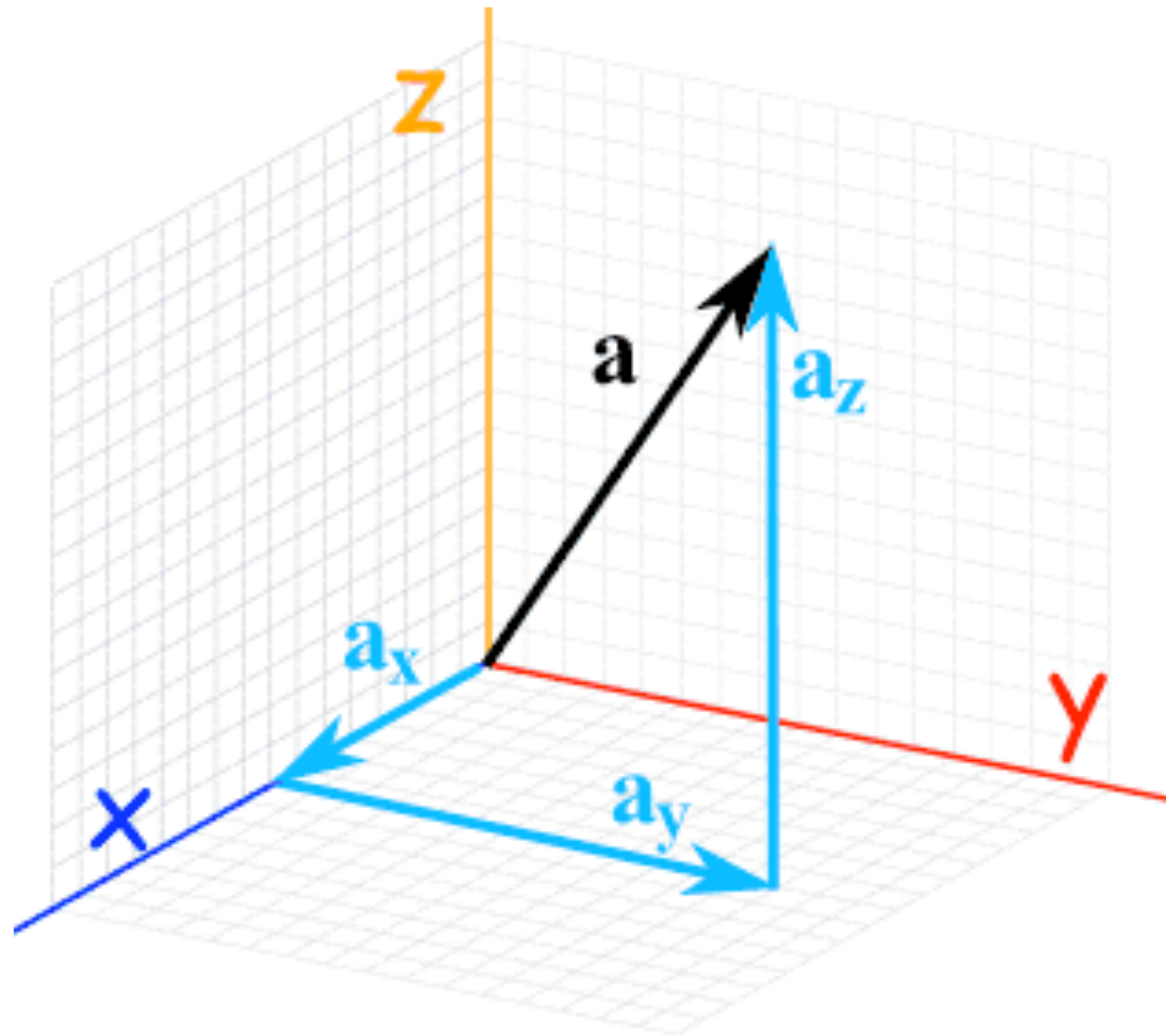


$$a = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

```
var a = [ [2],  
          [3] ];
```



# 3D Vector



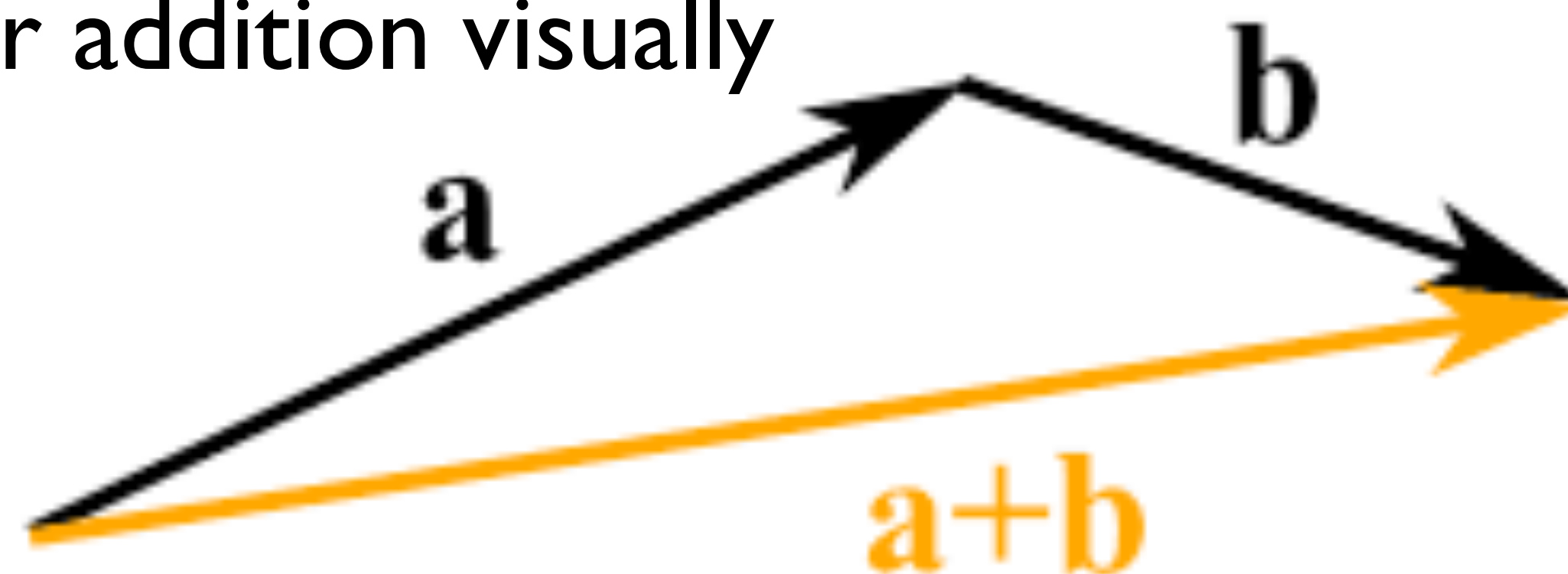
$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

```
var a = [ [ax],  
          [ay],  
          [az] ];
```

# Vector Addition and Subtraction

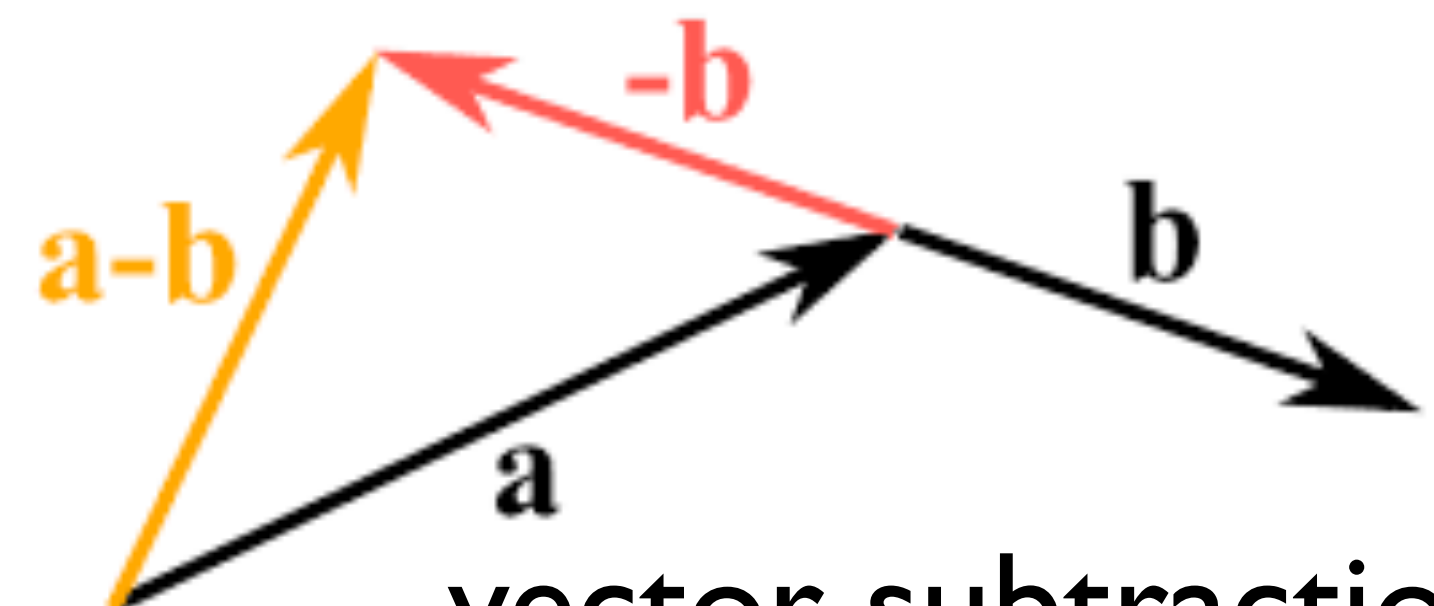
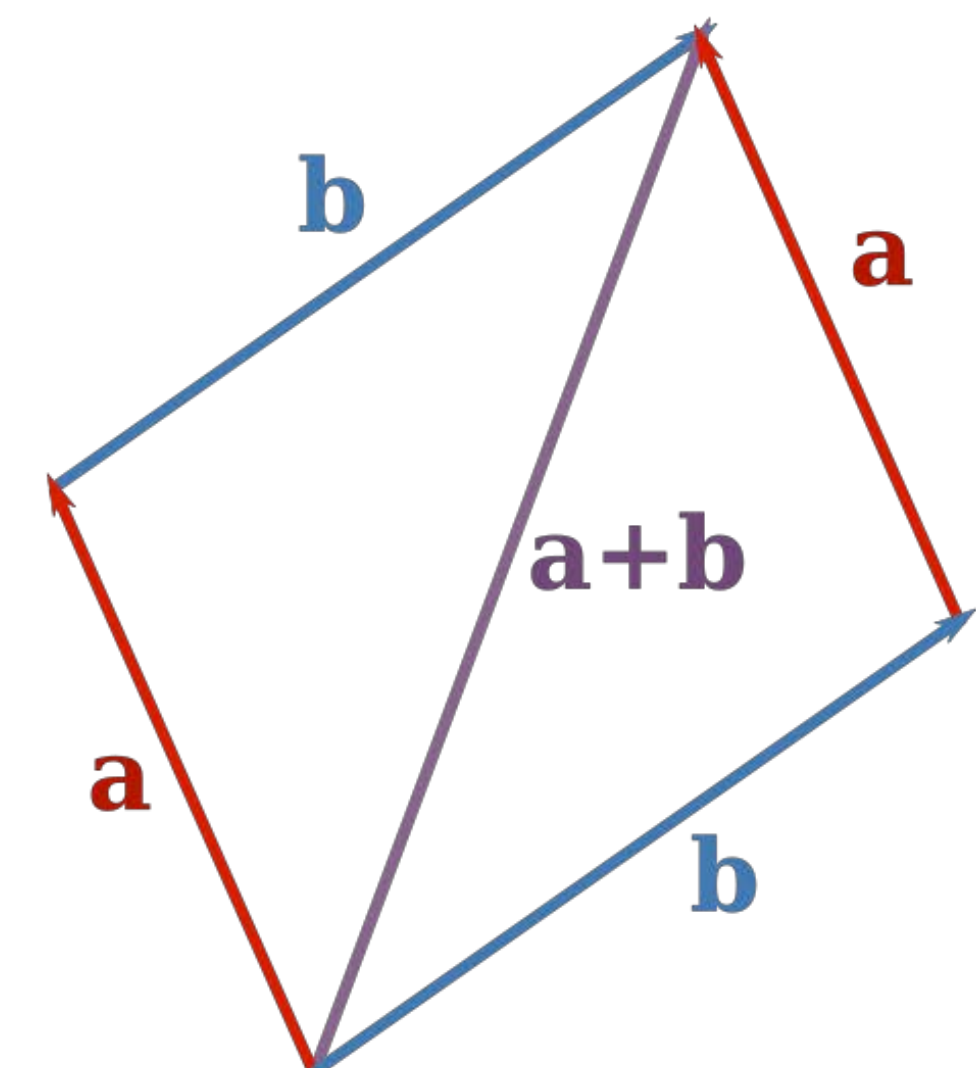
$$a + b = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$$

vector addition visually



vector result

vector addition is order independent



vector subtraction is addition with negated vector





# Magnitude and Unit Vector

The magnitude of a vector is the square root of the sum of squares of its components

$$\|a\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

A unit vector has a magnitude of one. Normalization scales a vector to unit length.

$$\hat{a} = \frac{a}{\|a\|}$$

A vector can be multiplied by a scalar

$$sa = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$$



scalar  
result

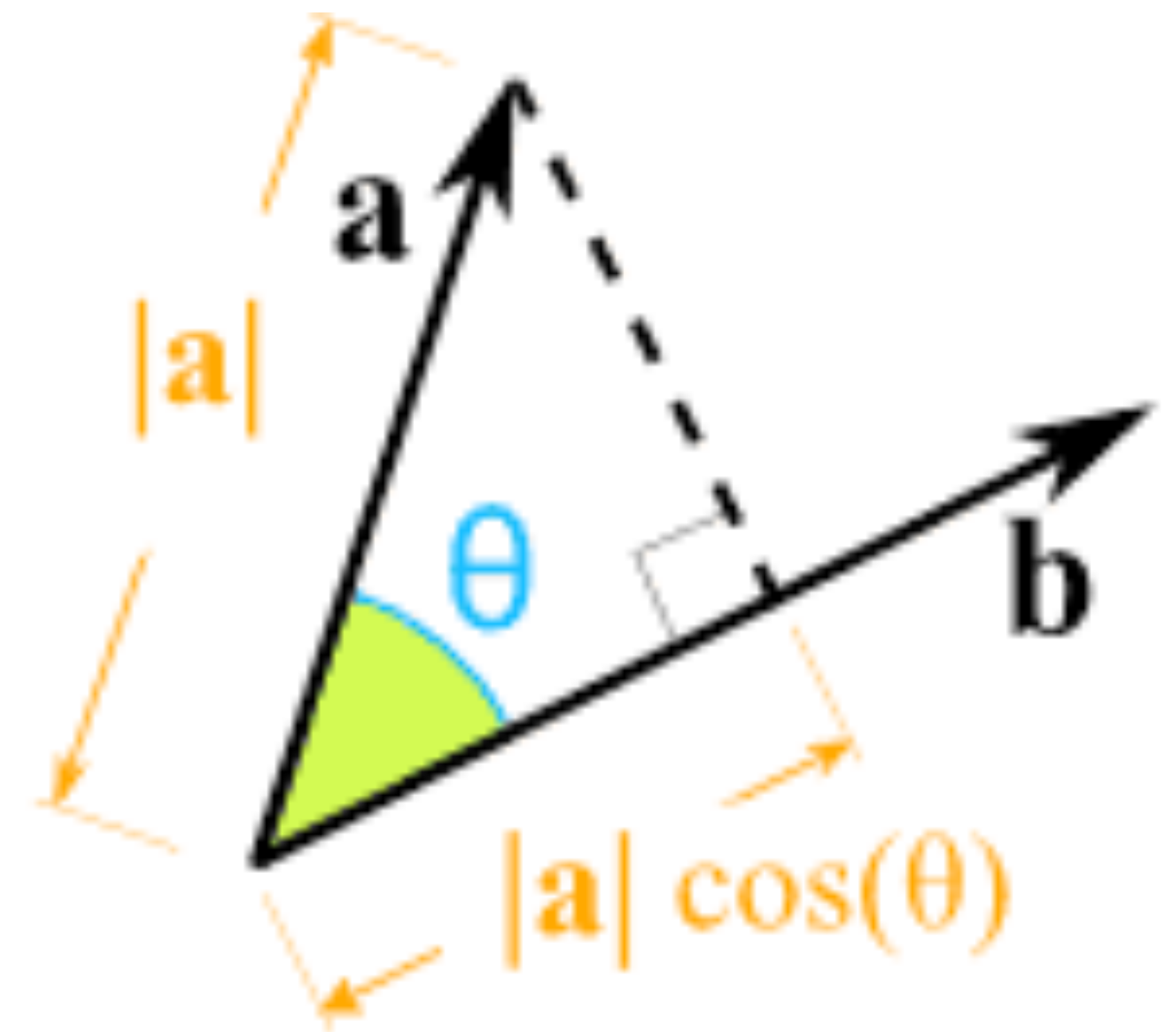


# Dot Product

$$\begin{aligned} a \bullet b &= a_x b_x + a_y b_y + a_z b_z \\ &= \|a\| \|b\| \cos(\theta) \end{aligned}$$

Measures the similarity in direction of  
two vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 * 3 + 1 * 2 = 8$$





# Projections

Dot products related to projections onto vectors.

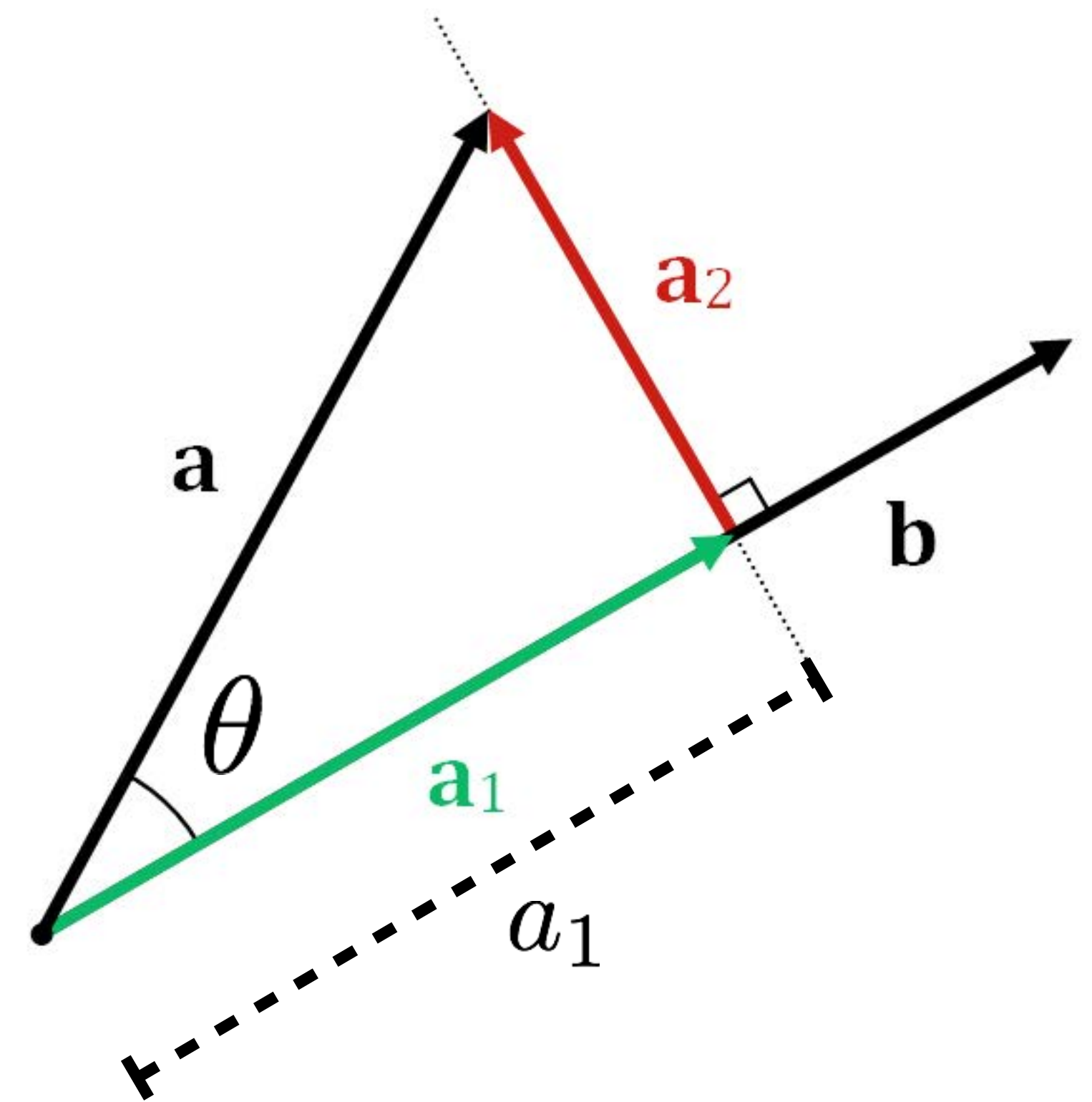
Scalar projection of one vector onto another

$$a_1 = |\mathbf{a}| \cos \theta = \mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$$

Vector projection

$$\mathbf{a}_1 = a_1 \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$  is unit vector



# Checkpoint

- What is the dot product of a vector with itself?

[Redacted]

- What is the dot product of two orthogonal vectors?

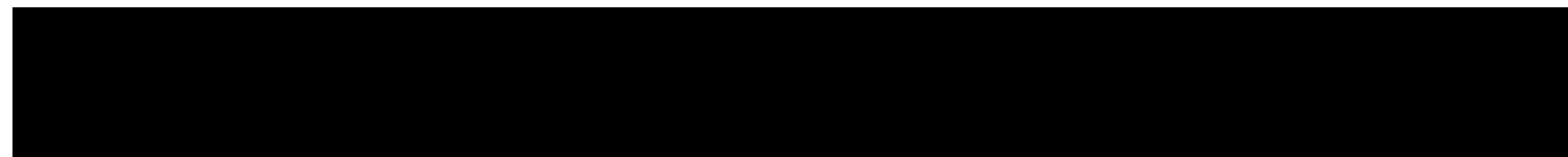
[Redacted]





# Checkpoint

- What is the dot product of a vector with itself?
  - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?



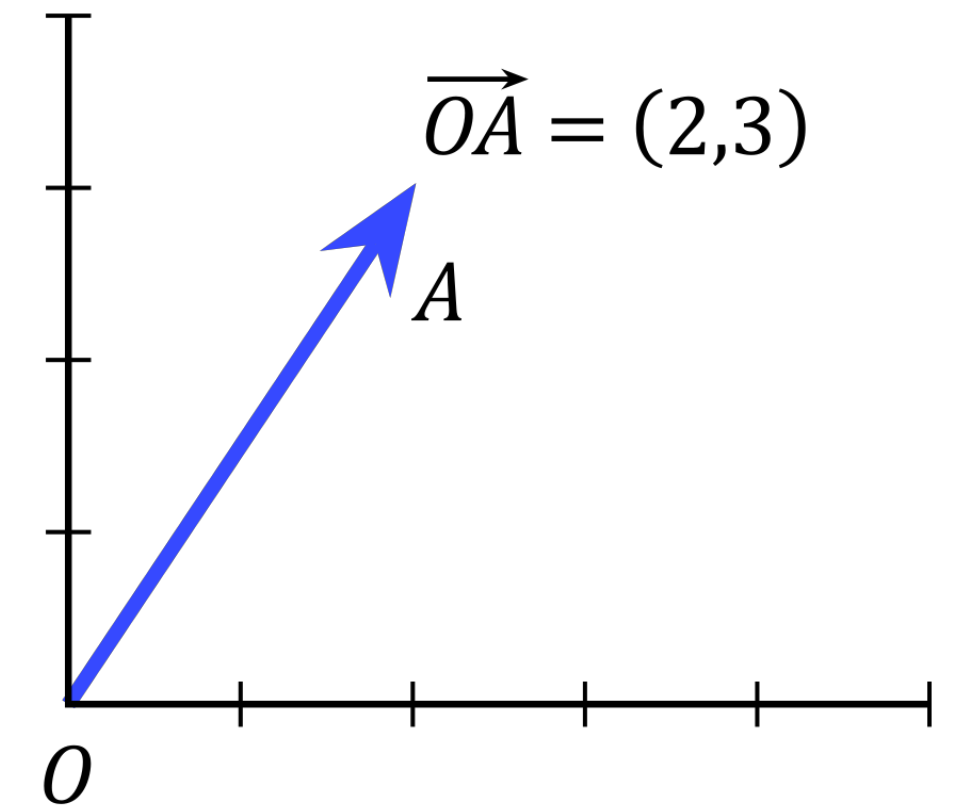
# Checkpoint

- What is the dot product of a vector with itself?
  - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?
  - 0





# Checkpoint

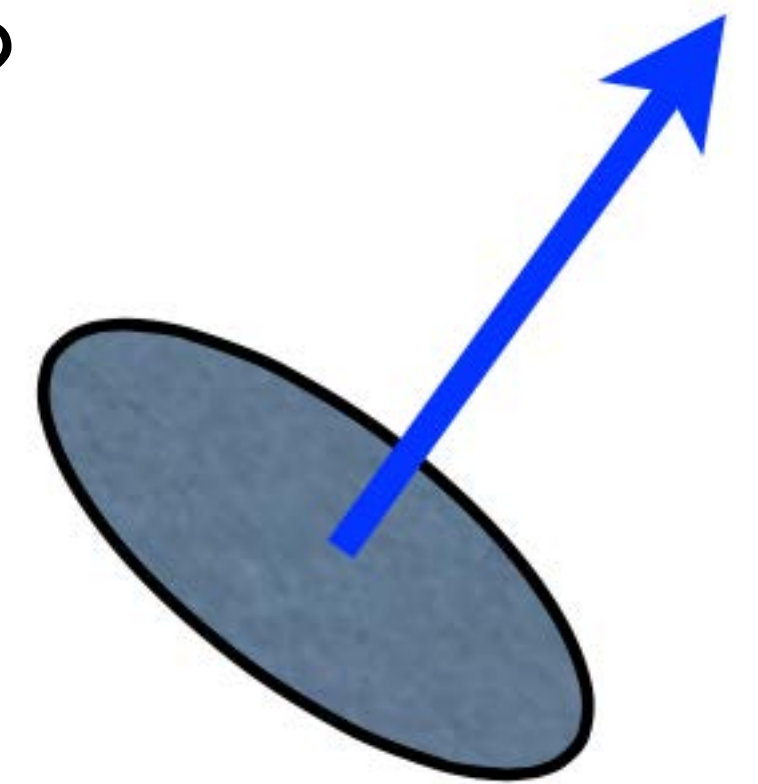


- How many unit vectors are perpendicular to a 2D vector?

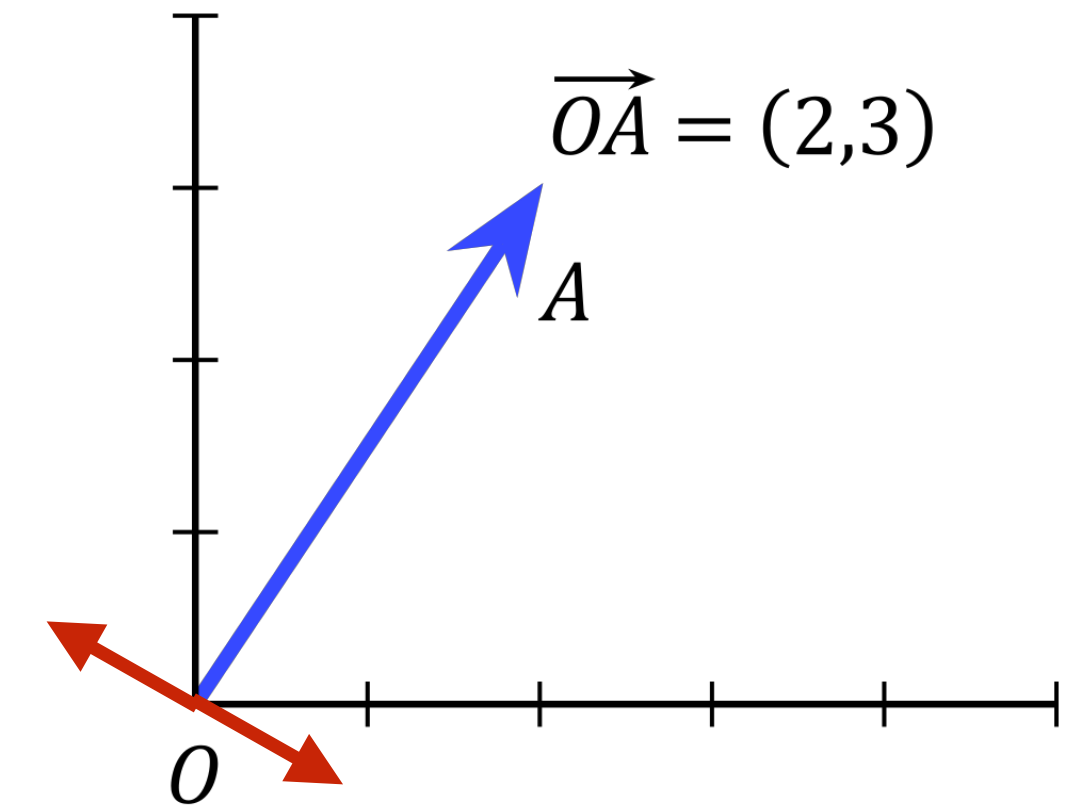
[Redacted]

- How many unit vectors are perpendicular to a 3D vector?

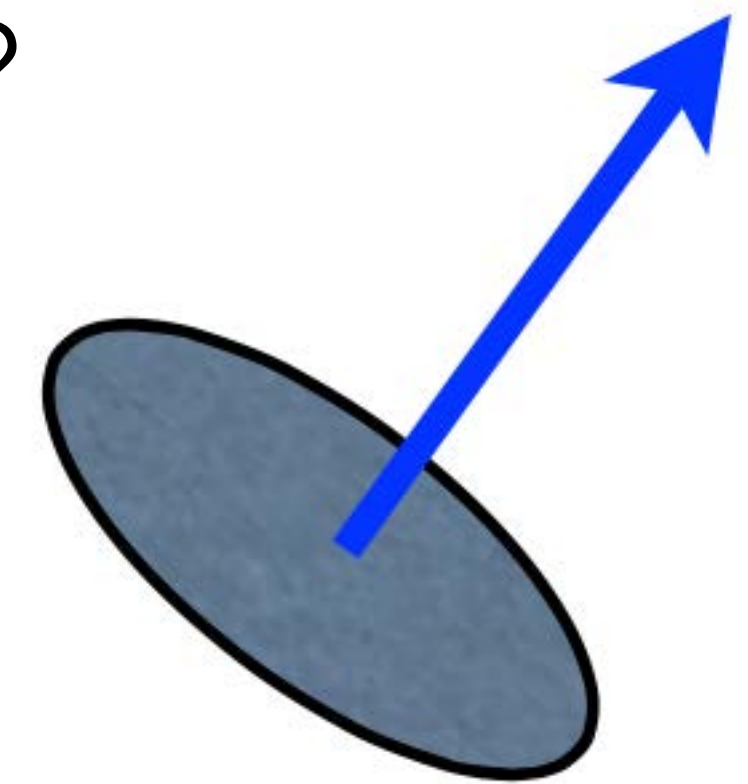
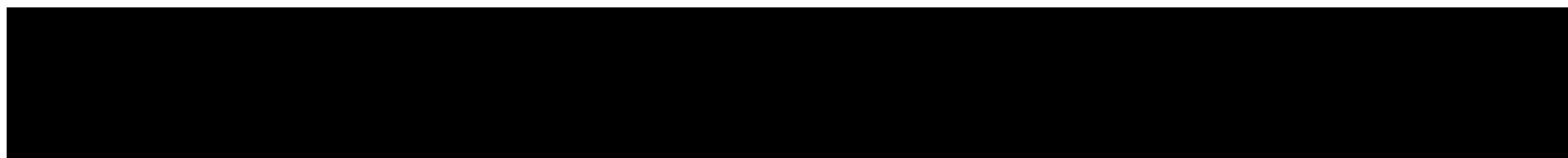
[Redacted]



# Checkpoint

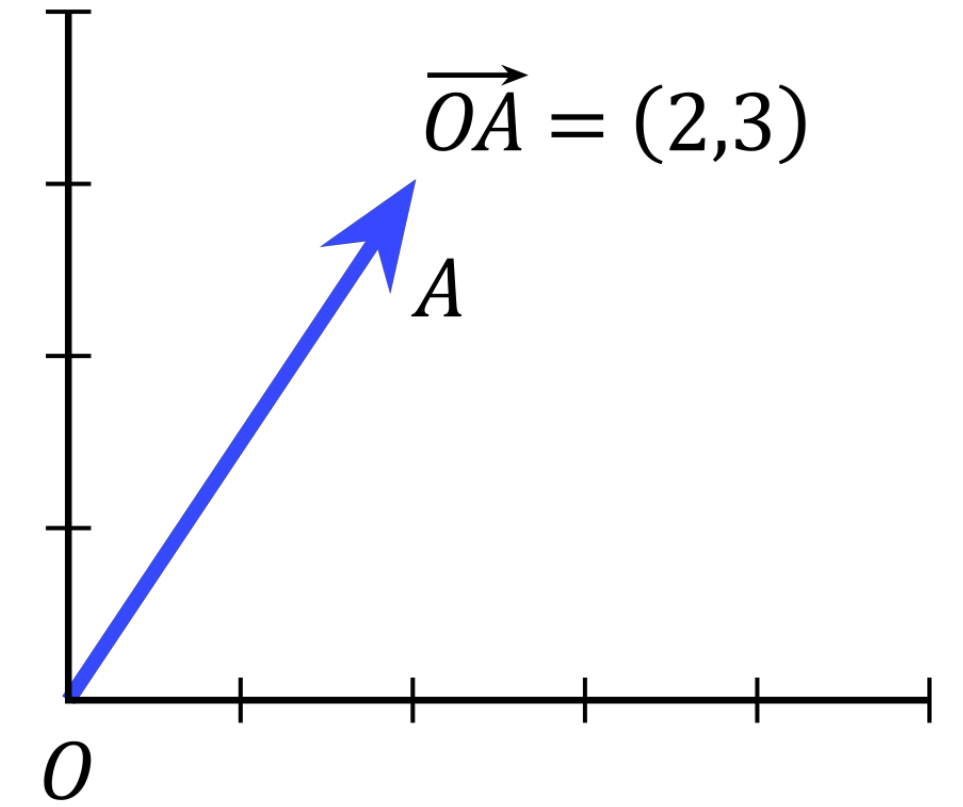


- How many unit vectors are perpendicular to a 2D vector?
  - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?

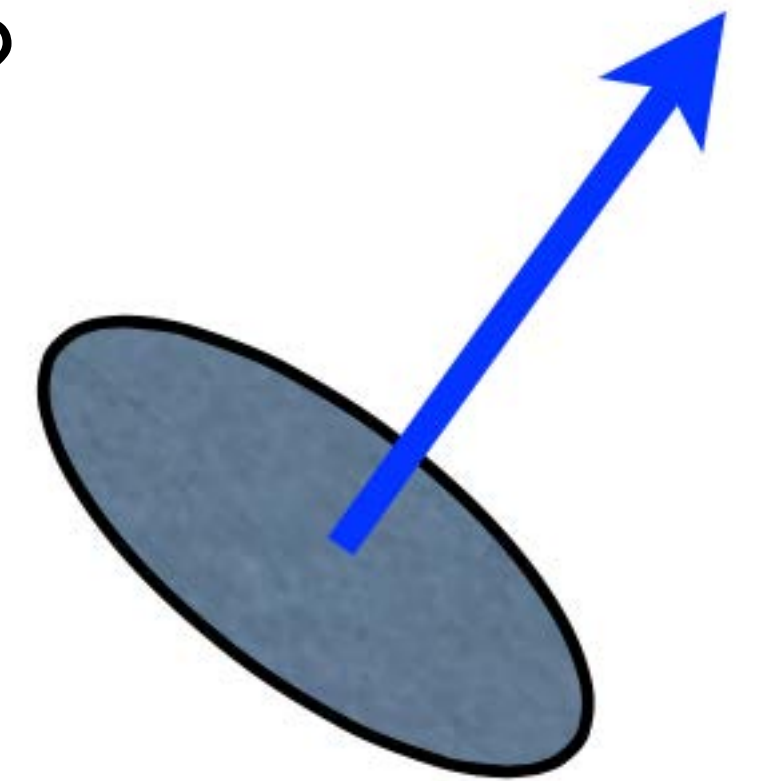




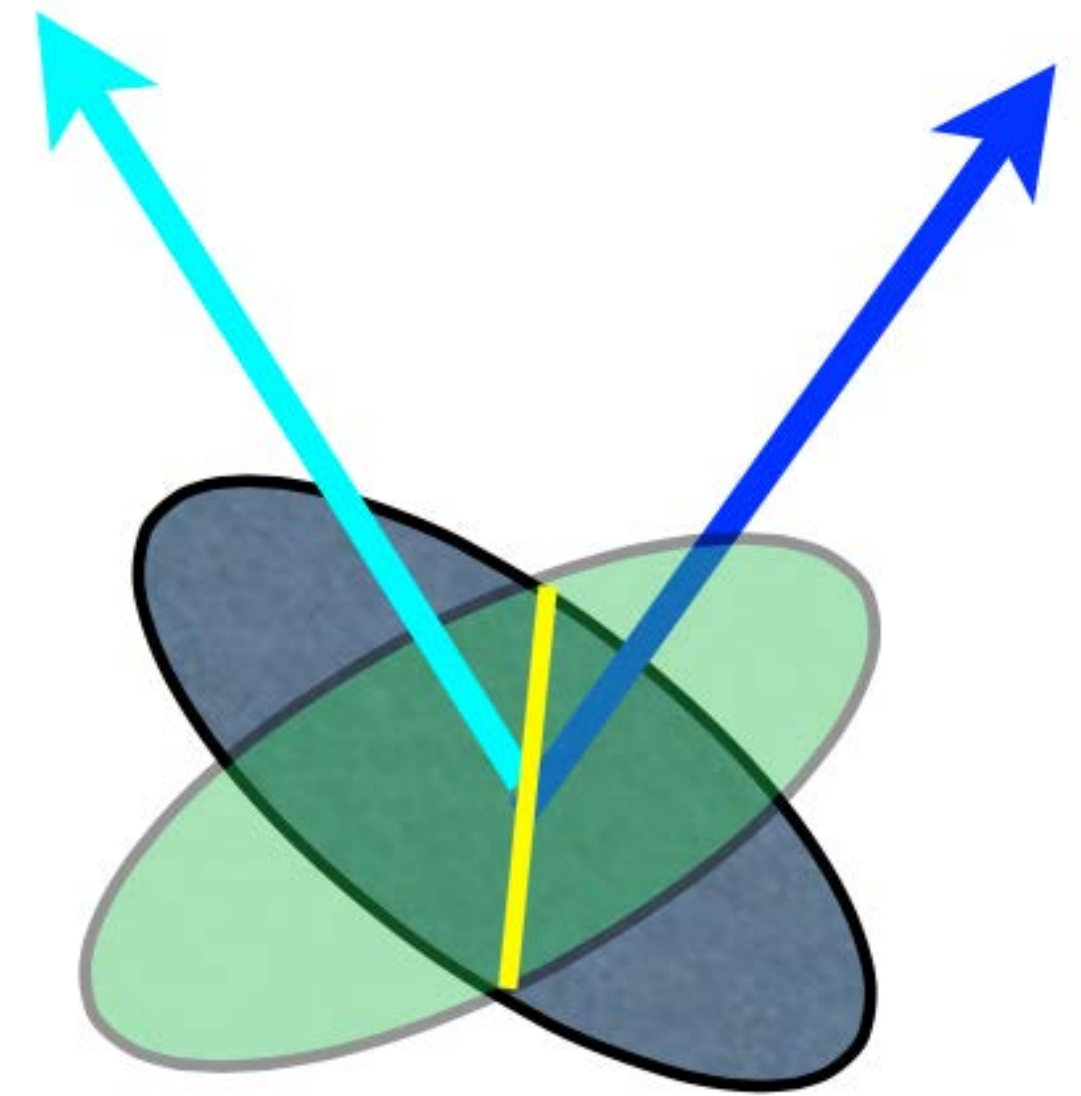
# Checkpoint



- How many unit vectors are perpendicular to a 2D vector?
  - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?
  - Infinite and lie in plane



Given two vectors, how to compute  
a vector orthogonal to both?





Assumes **a** and **b** are in same frame

# Cross Product

$$c_x = a_y b_z - a_z b_y$$

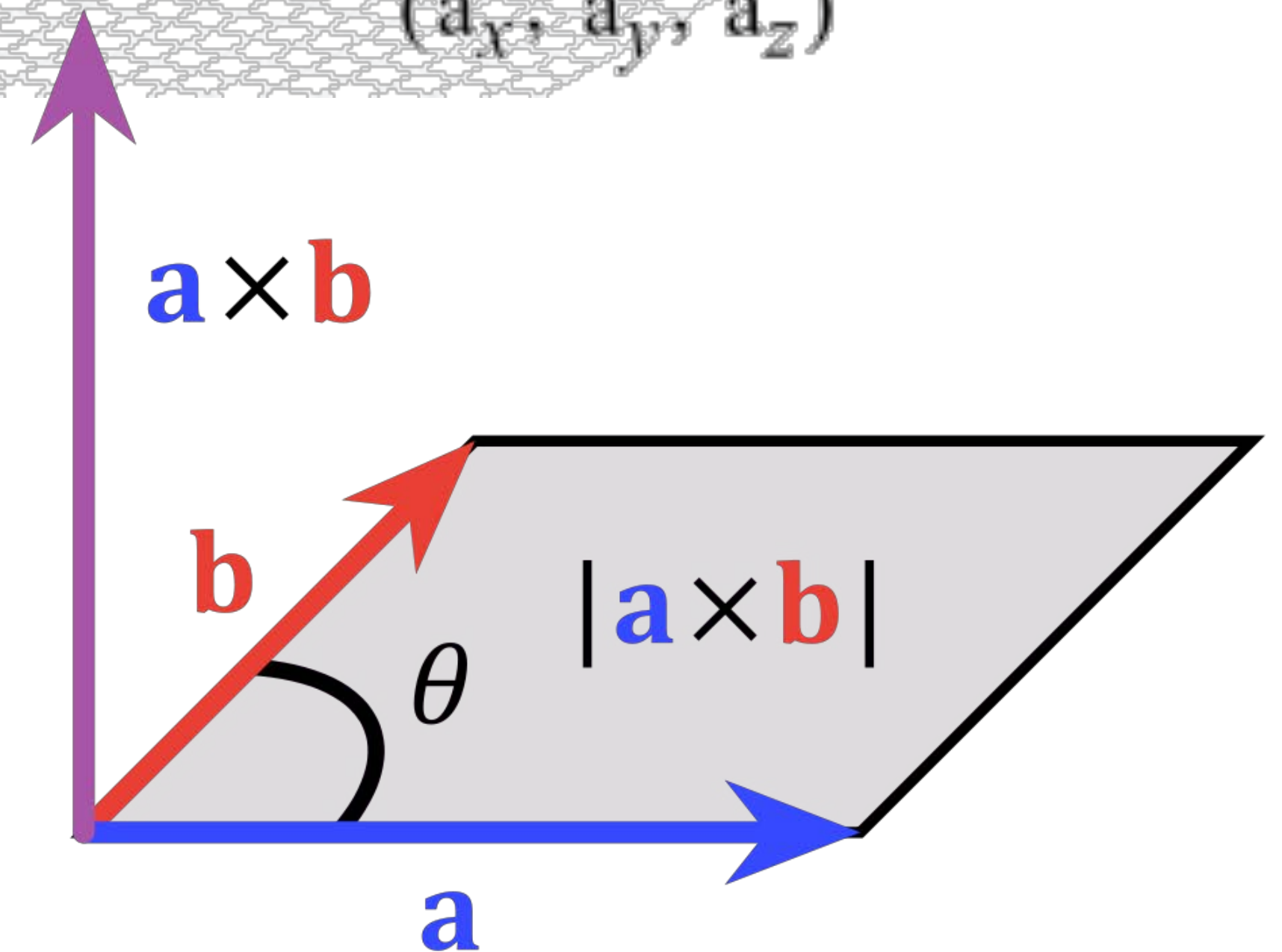
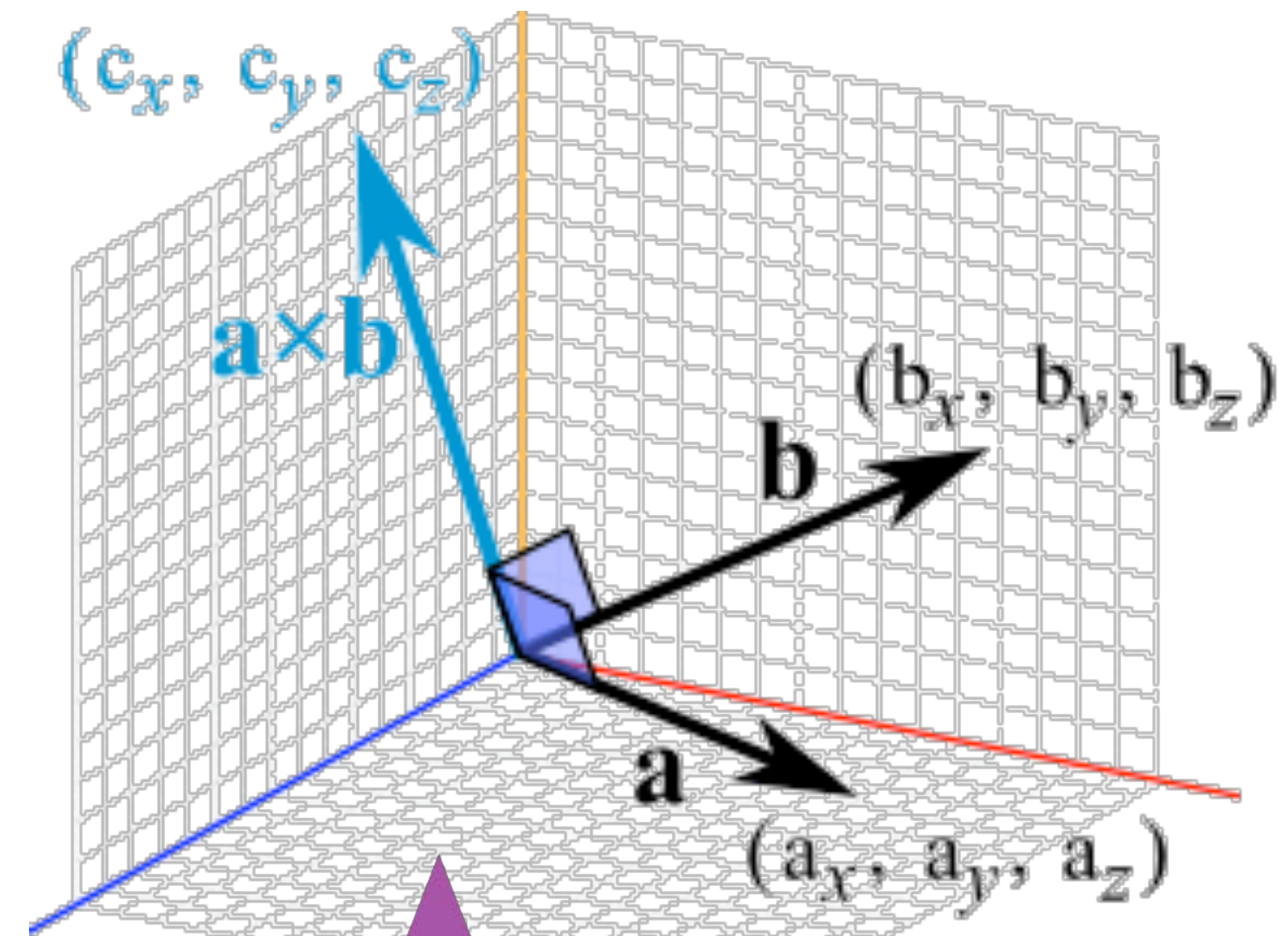
$$c_y = a_z b_x - a_x b_z$$

$$c_z = a_x b_y - a_y b_x$$

Results in new vector **c** orthogonal to both original vectors **a** and **b**

Length of vector **c** is equal to area of parallelogram formed by **a** and **b**

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| \sin \theta$$



# Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [  
  [1, 0, 0, 0],  
  [0, 1, 0, 0],  
  [0, 0, 1, 0],  
  [0, 0, 0, 1] ];
```

What is this  
matrix?



# Matrix-vector multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$





# For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} =$$



For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$



# Matrix-vector multiplication

(two interpretations)

1) **Row story:** dot product of each matrix row

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

2) **Column story:** linear combination of matrix columns

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} j + \begin{bmatrix} b \\ e \\ h \end{bmatrix} k + \begin{bmatrix} c \\ f \\ i \end{bmatrix} l$$





# Revisiting the cross product: Skew-symmetric matrices

A given 3D vector  $\mathbf{a} = (a_1 \ a_2 \ a_3)^T$

can be expressed as a skew-symmetric matrix

$$[\mathbf{a}]_{\times} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

such that the cross product with another vector is a matrix multiplication

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_{\times} \mathbf{b}$$





# Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [  
  [1, 0, 0, 0],  
  [0, 1, 0, 0],  
  [0, 0, 1, 0],  
  [0, 0, 0, 1] ];
```

```
var mat = [  
  [1, 0, 0, tx],  
  [0, 1, 0, ty],  
  [0, 0, 1, tz],  
  [0, 0, 0, 1] ];
```

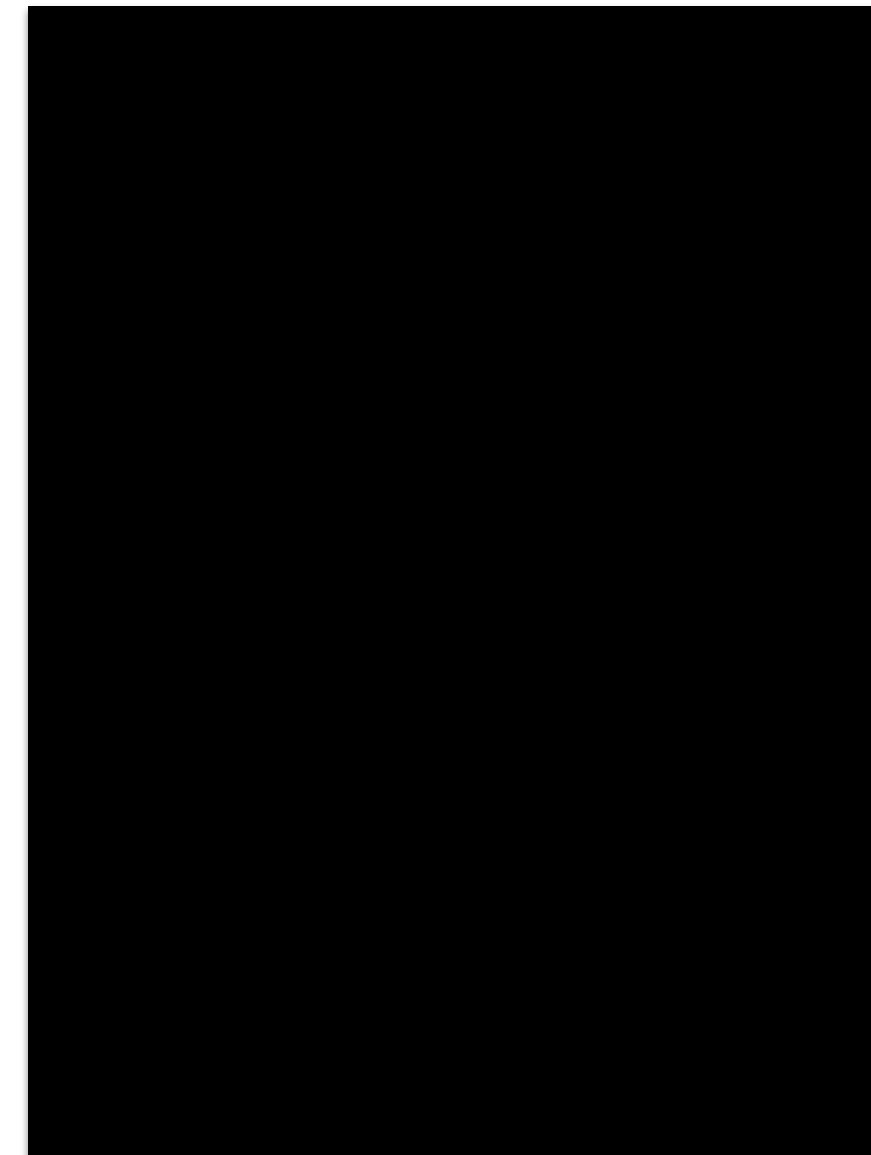
What is this  
matrix?





# Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} =$$



# Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$



# Matrix Multiplication

- Scalar Multiplication

$$\lambda \mathbf{A} = \lambda \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1m} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda A_{n1} & \lambda A_{n2} & \cdots & \lambda A_{nm} \end{pmatrix}.$$

- Multiplication of two matrices

$$(\mathbf{AB})_{ij} = \sum_{k=1}^m A_{ik} B_{kj}.$$

Each entry of product matrix  $\mathbf{AB}$  is a dot product of a row of  $\mathbf{A}$  with a column of  $\mathbf{B}$

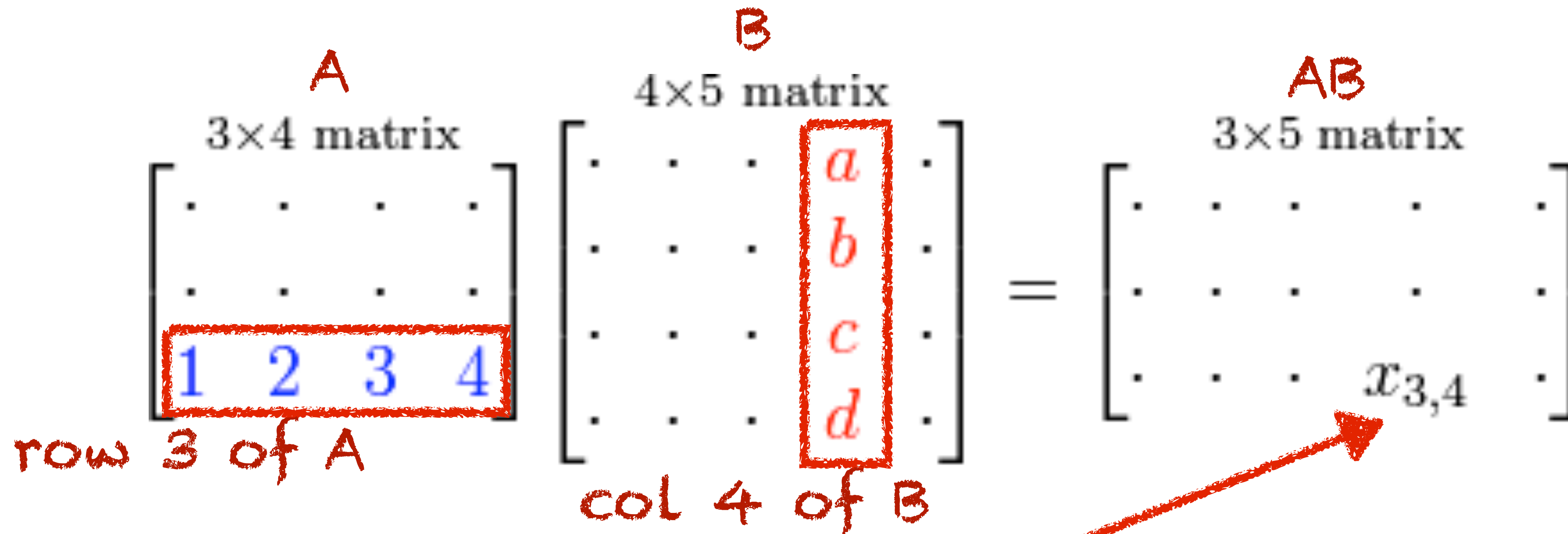




# Matrix multiplication

Finger sweeping rule should be second nature!

- Left finger sweeps left to right
- Right finger sweeps top to bottom



Do this dot product for each row/column combination

$$\begin{aligned} x_{3,4} &= (\mathbf{1, 2, 3, 4}) \cdot (\mathbf{a, b, c, d}) \\ &= \mathbf{1} \times \mathbf{a} + \mathbf{2} \times \mathbf{b} + \mathbf{3} \times \mathbf{c} + \mathbf{4} \times \mathbf{d} \end{aligned}$$

# Matrix Multiplication Reminders

- Number of columns of  $A$  must match number of rows of  $B$
- Multiplying a  $(M \times K)$  matrix with a  $(K \times N)$  matrix will produce an  $(M \times N)$  matrix
- Matrix multiplication is not commutative:  $AB \neq BA$



# Example

"Dot Product"

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 \\ \end{bmatrix}$$

$$(1, 2, 3) \cdot (7, 9, 11) = 1 \times 7 + 2 \times 9 + 3 \times 11 = 58$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \end{bmatrix}$$

$$(1, 2, 3) \cdot (8, 10, 12) = 1 \times 8 + 2 \times 10 + 3 \times 12 = 64$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$$(4, 5, 6) \cdot (7, 9, 11) = 4 \times 7 + 5 \times 9 + 6 \times 11 = 139$$

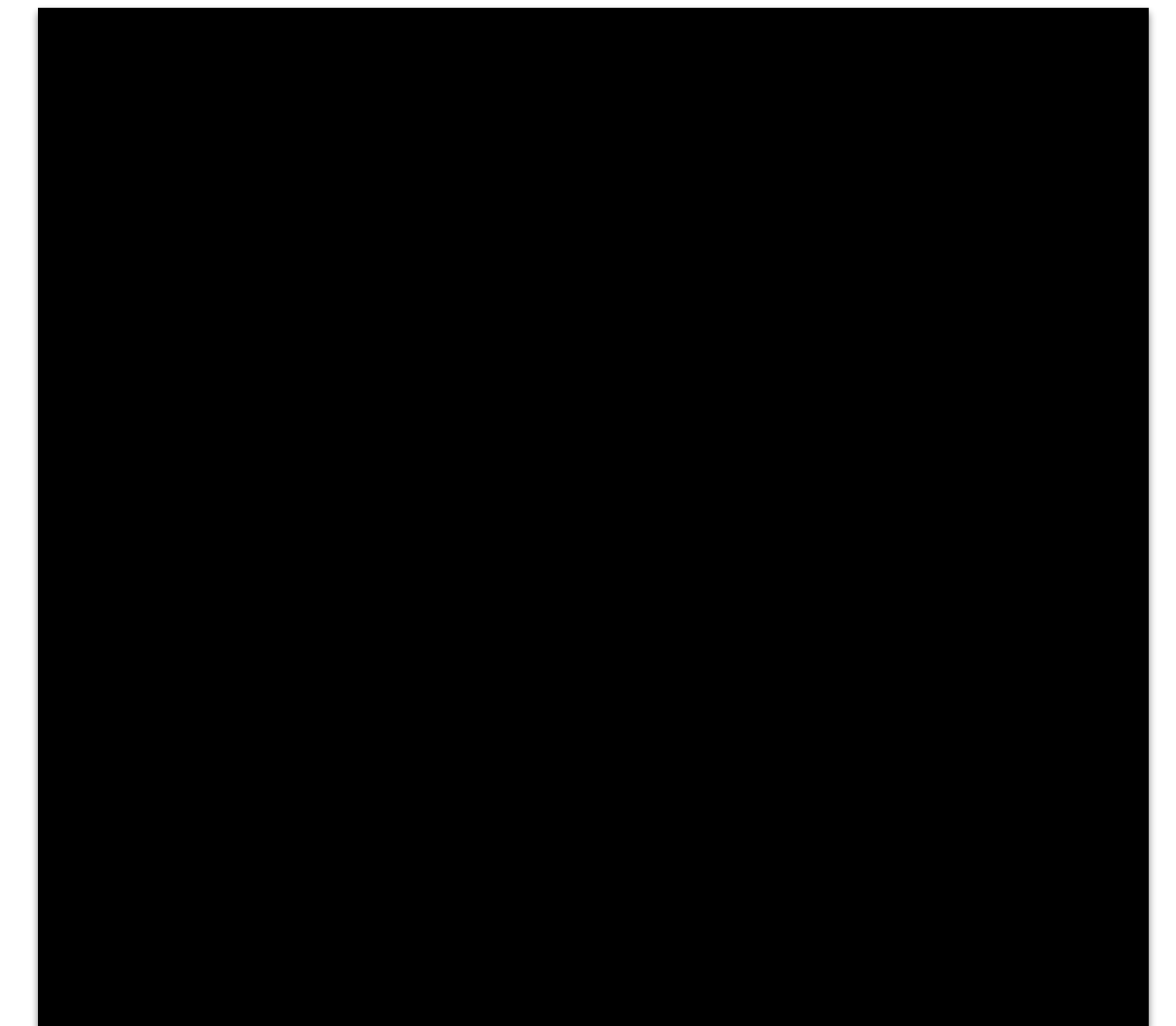
$$(4, 5, 6) \cdot (8, 10, 12) = 4 \times 8 + 5 \times 10 + 6 \times 12 = 154$$





For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} =$$



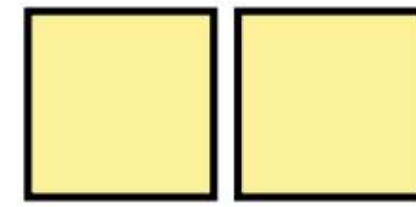
# For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 2 & 5 \end{bmatrix}$$

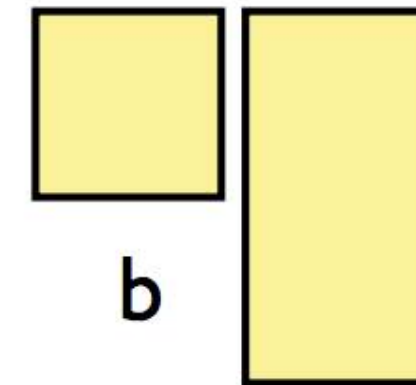


# Checkpoint

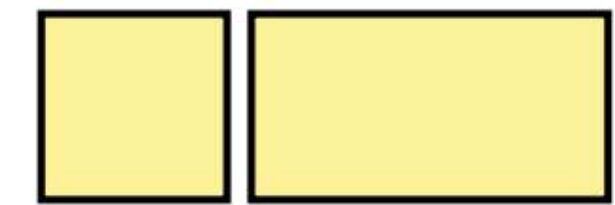
- Which of the following matrix multiplications are valid?



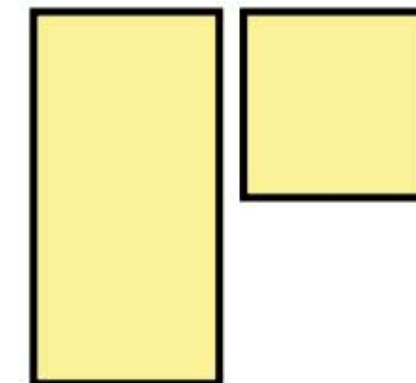
a



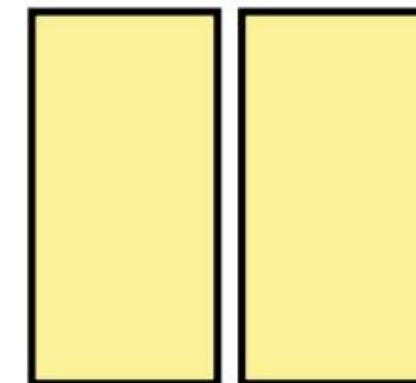
b



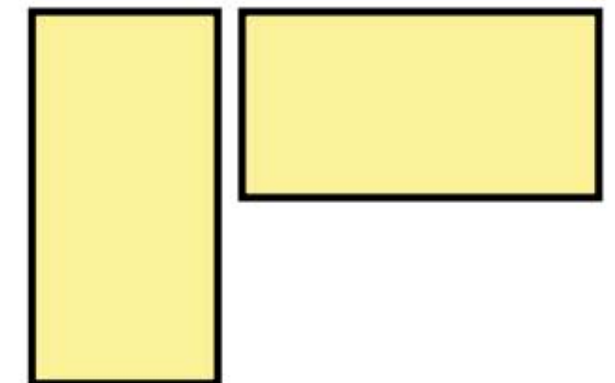
c



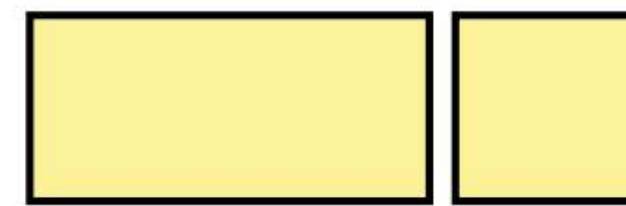
d



e



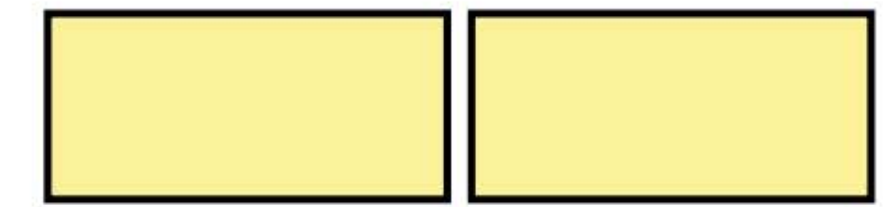
f



g



h

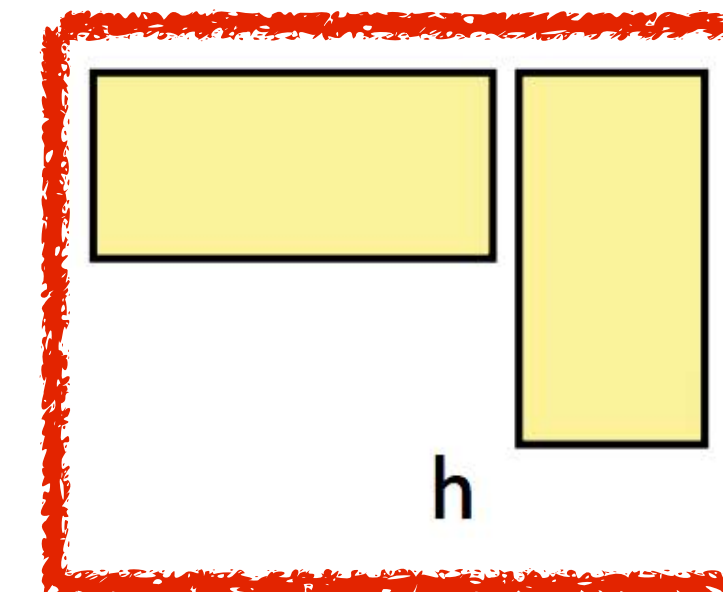
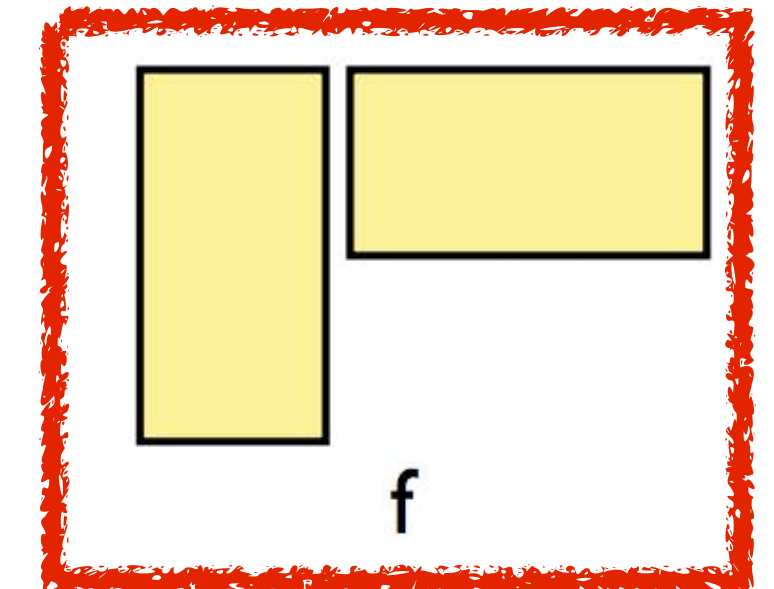
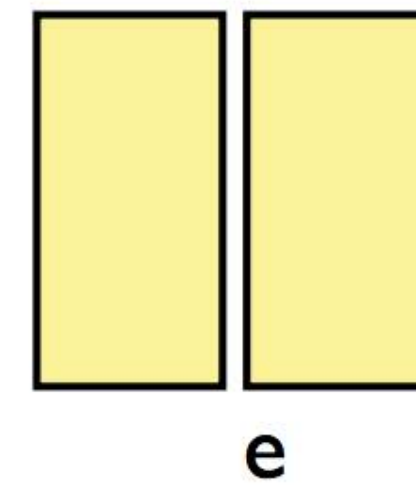
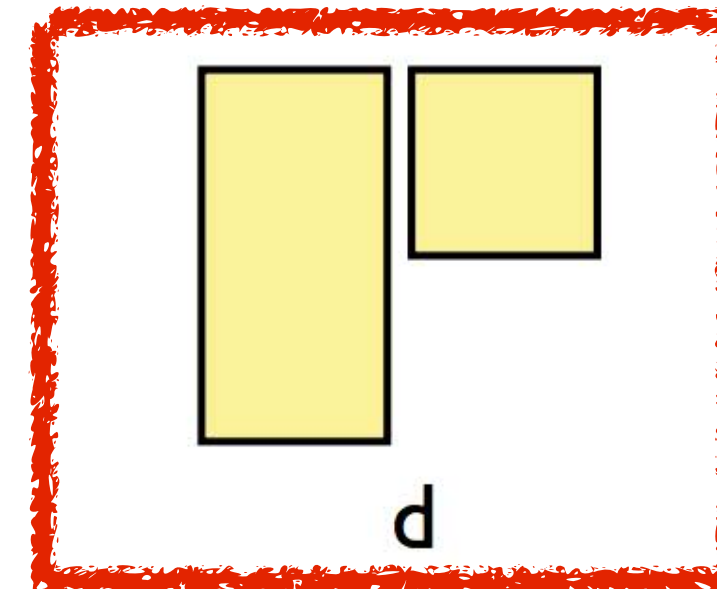
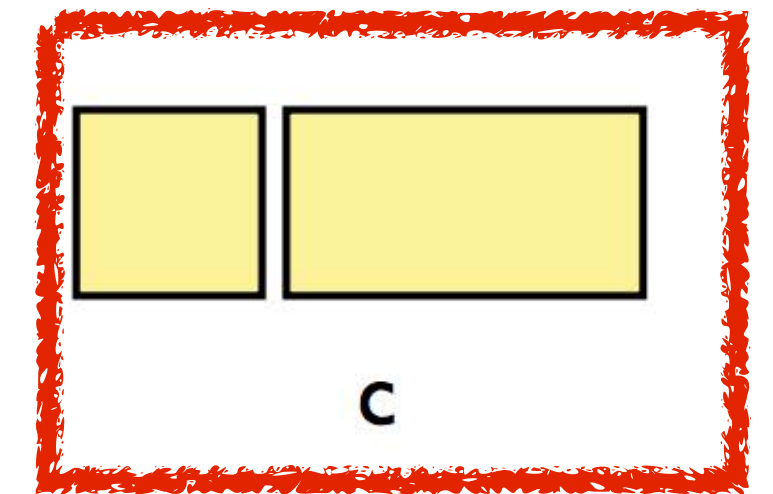
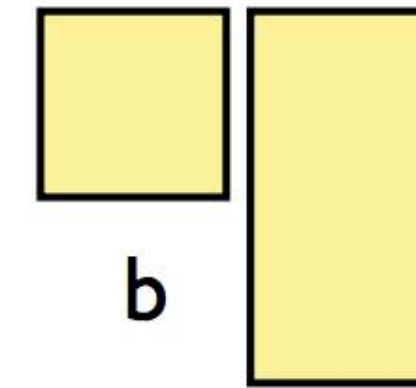
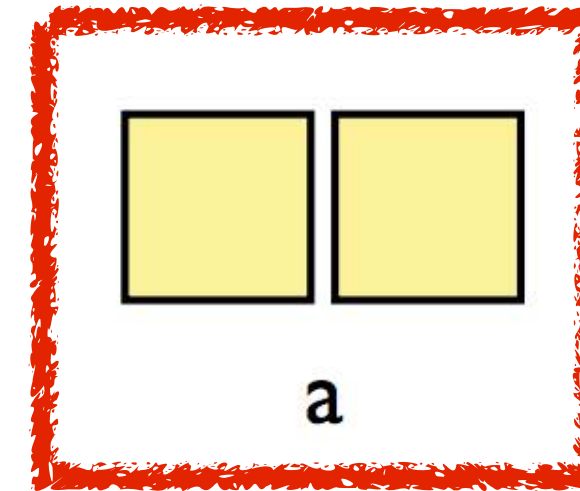


i



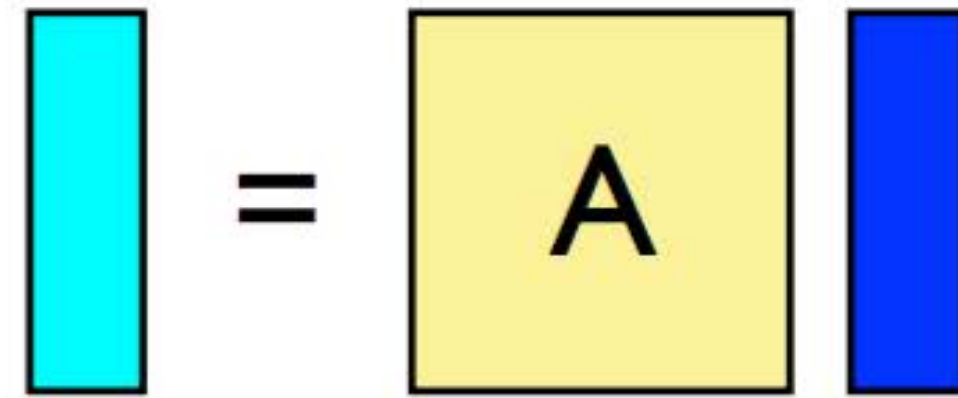
# Checkpoint

- Which of the following matrix multiplications are valid?



# Matrices as projections

- Matrix multiplication projects from one space to another.


$$\text{Cyan Bar} = A \text{ Blue Bar}$$

Data projected into new  
coordinate system

Data in original  
coordinate system

# Notable Matrices and Operations

- Matrix identity ( $I$ ) causes no change:  $A = I_m A = A I_n$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal elements  $A_{ij} = 1$
- Off-diagonal elements  $A_{ij} = 0, i \neq j$
- Matrix inverse ( $A^{-1}$ ): if  $AA^{-1} = A^{-1}A = I$
- Distributing matrix inverse:  $(AB)^{-1} = B^{-1}A^{-1}$

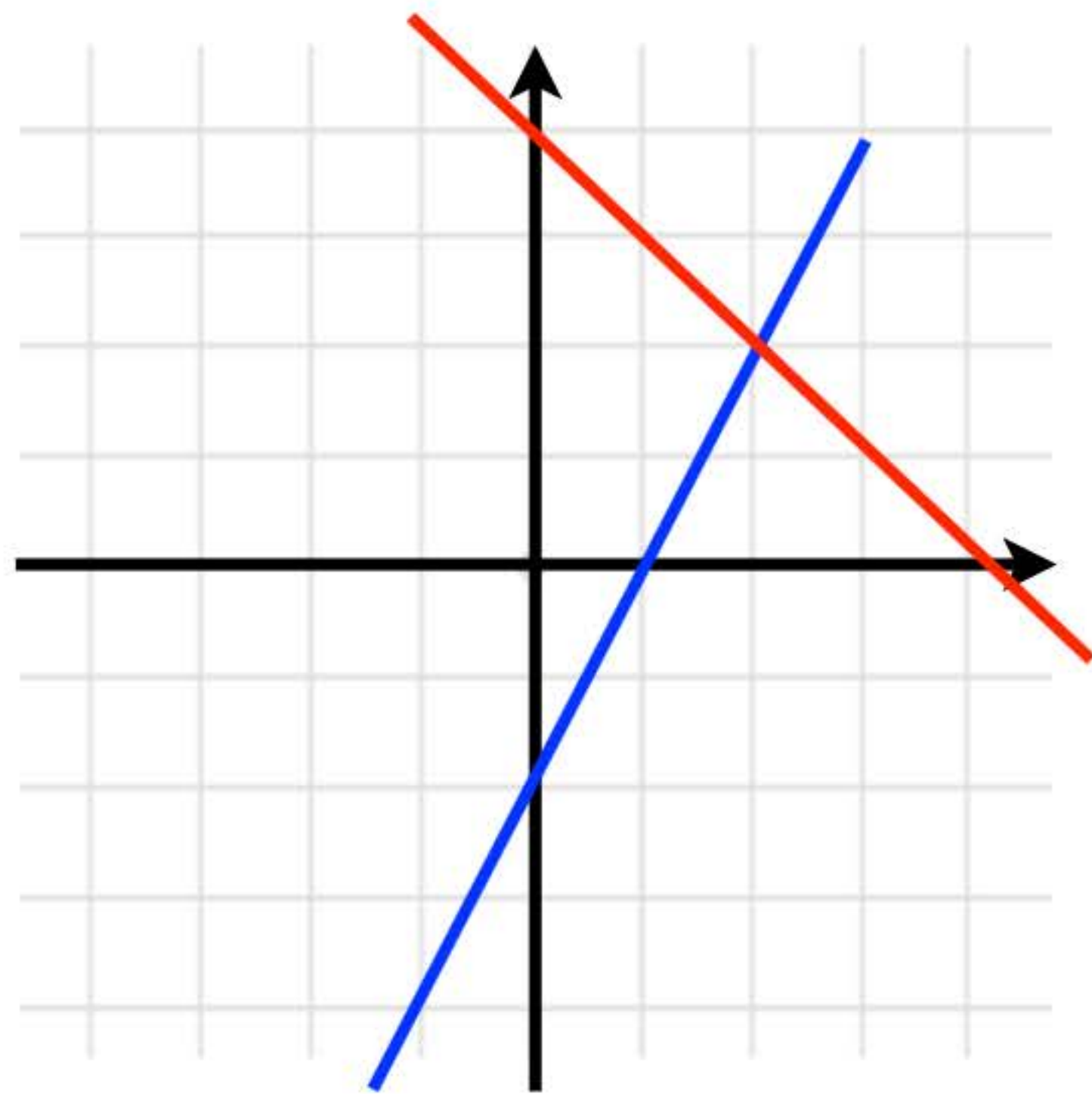
- Matrix transpose ( $A^T$ ): a matrix's reflection about its diagonal

- Distributing matrix transpose:  $(AB)^T = B^T A^T$
- $$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$





# Matrix Geometry: Row Story



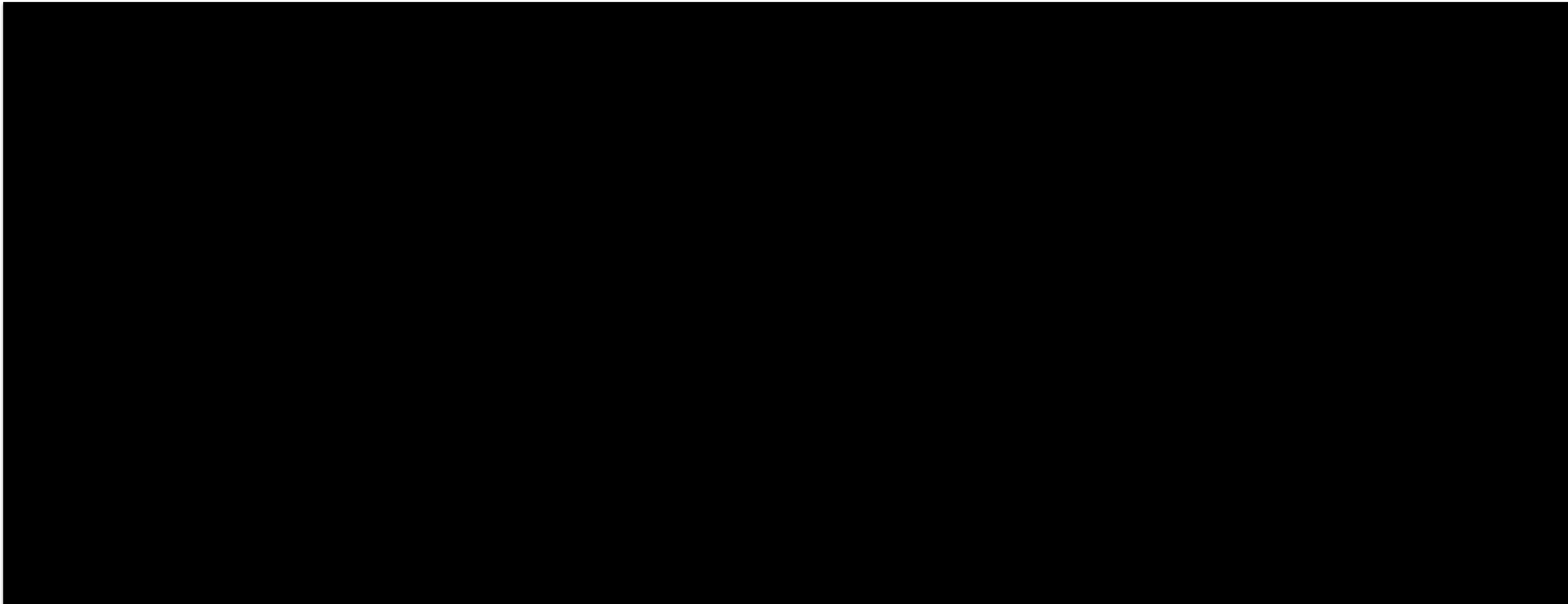
- Each row of a linear system represents a hyperplane. (In 2D, that's also a line!)
- The solution to the system is the intersection of those hyperplanes

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

# Solving linear systems

What would be the direct way to solve for  $\mathbf{x}$ ?

$$A\mathbf{x} = \mathbf{b}$$



# Solving linear systems

What would be the direct way to solve for  $\mathbf{x}$ ?

$$\mathbf{Ax} = \mathbf{b}$$

Invert  $\mathbf{A}$  and multiply by  $\mathbf{b}$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$





# Matrix rank and inversion

- Let  $A$  be a square  $n$  by  $n$  matrix.  $A$  is invertible if full rank and a matrix  $B$  exists such that
- Rank of a matrix  $A$  is the size of the largest collection of linearly independent columns of  $A$
- $A$  is invertible (nonsingular) if it has full rank
- Gaussian elimination can find matrix inverse
- Singular matrix cannot be inverted this way

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

$$[A|I] = \left[ \begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array} \right]$$

$$[I|B] = \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array} \right]$$



# Solution by Decomposition

- In real applications, inverse not computed to solve linear systems
  - Efficiency, numerical precision, etc.

- Matrix decomposed into product of lower and upper triangular matrices

- LU decomposition  $\mathbf{A} = \mathbf{LU}$  
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

- Cholesky decomposition  $\mathbf{A} = \mathbf{LL}^T$

- Permits finding solution by forward substitution  $\mathbf{Ly} = \mathbf{b}$   
followed by backward substitution  $\mathbf{L}^T \mathbf{x} = \mathbf{y}$



# Solving linear systems

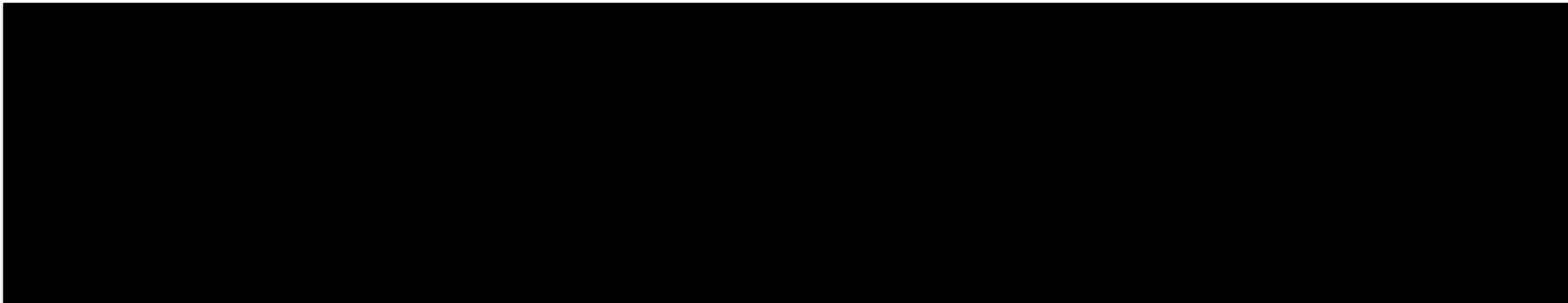
What would be the direct way to solve for  $\mathbf{x}$ ?

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$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

Can this always be done?





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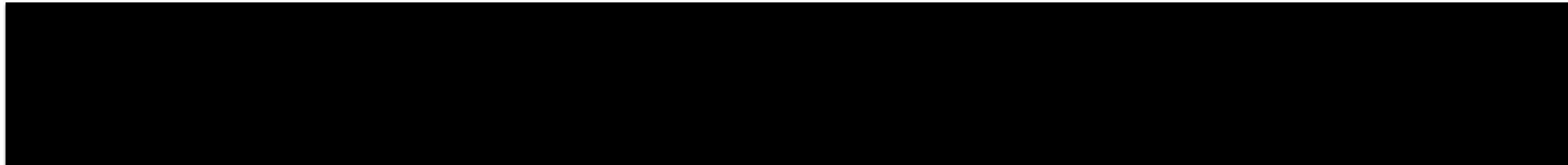
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Pseudoinverse least-squares approximation

$$\mathbf{x} = \mathbf{A}_{\text{left}}^+\mathbf{b}$$



# Pseudoinverse

- For matrix  $A$  with dimensions  $N \times M$  with full rank
- Find solution that minimizes squared error:  $\|Ax - b\|_2$
- Left pseudoinverse, for when  $N > M$ , (i.e., “tall”)

$$A_{\text{left}}^{-1} = (A^T A)^{-1} A^T \quad \text{s.t.} \quad A_{\text{left}}^{-1} A = I_n$$

- Right pseudoinverse, for when  $N < M$ , (i.e., “wide”)

$$A_{\text{right}}^{-1} = A^T (A A^T)^{-1} \quad \text{s.t.} \quad A A_{\text{right}}^{-1} = I_m$$





Next lecture:  
Representations I:  
Transformations

