Lecture 02

Planning - I - Path Planning

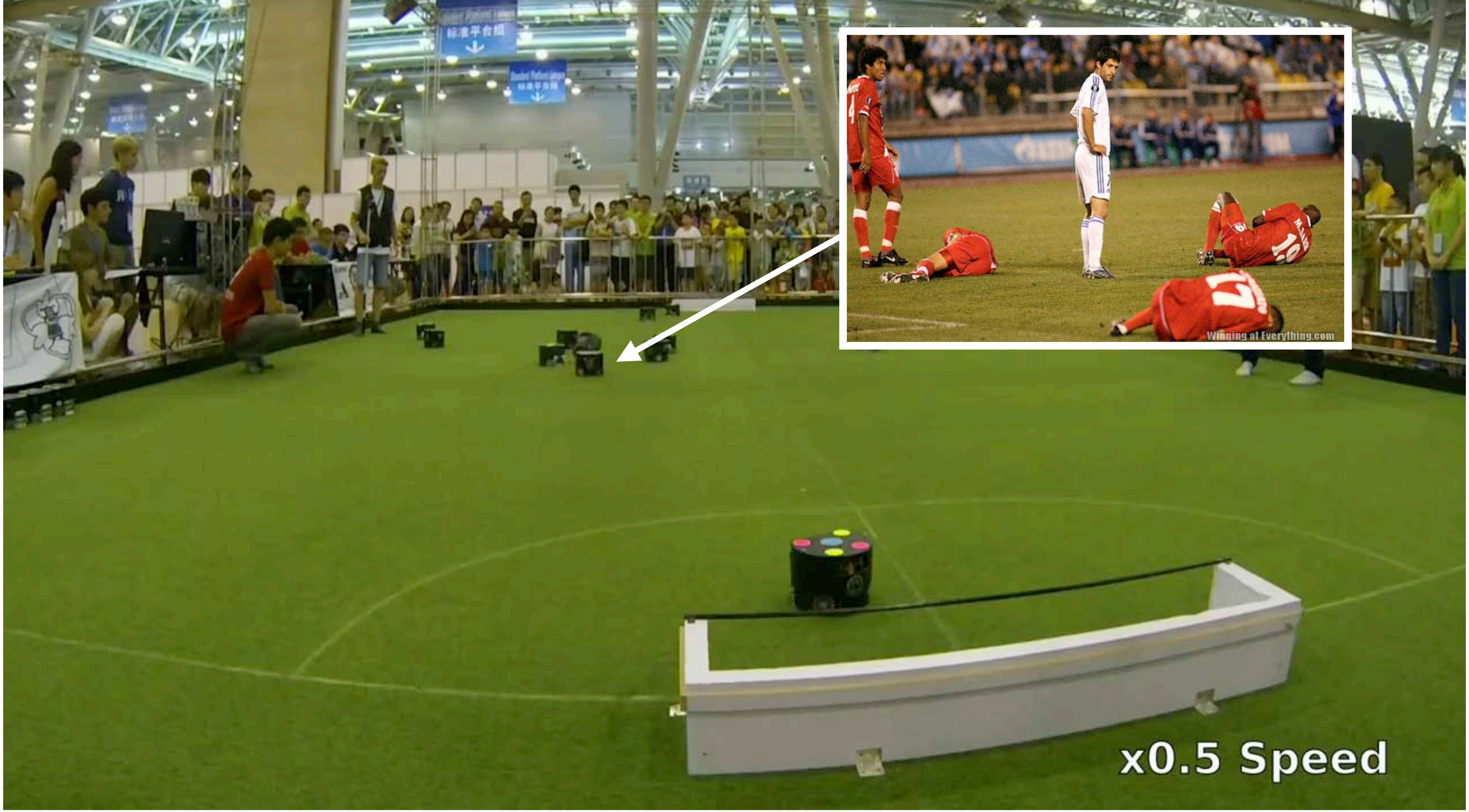
Course Logistics

- Quiz 1 will be released tomorrow evening 6pm on Gradescope and will be due on 01/29 12pm (before the Wed Lecture)
  - Quiz will be released every week at 6pm on Tuesdays and will be due at 12pm on Wednesdays.
  - You are allowed to refer the course material to answer them.
  - You can discuss the quiz on Ed discussion after the due time.
  - Each Quiz will have 2 questions for 0.5 pts each.
  - They are designed to be answered in less than 5 mins each.
  - When you start the quiz, you will have 20 mins to answer them.
  - Best 10 quizzes out of 12 will be used for final grades.
  - Use of AI tools is **NOT PERMITTED.**
- Project 1 will be posted on 01/29 and will be due 02/05
  - Start early!
- EdStem - I will add all the students to the discussion board by today evening and send an announcement.
  - Note: Starting tomorrow, all the announcements will be via Ed and **NOT Canvas**

Path Planning

CMDragons 2015 Pass-ahead Goal

CMDragons 2015 slow-motion multi-pass goal



CMDragons 2015 slow-motion multi-pass goal

Slide borrowed from Michigan Robotics autorob.org

http://www.cs.cmu.edu/~coral/projects/cobot/

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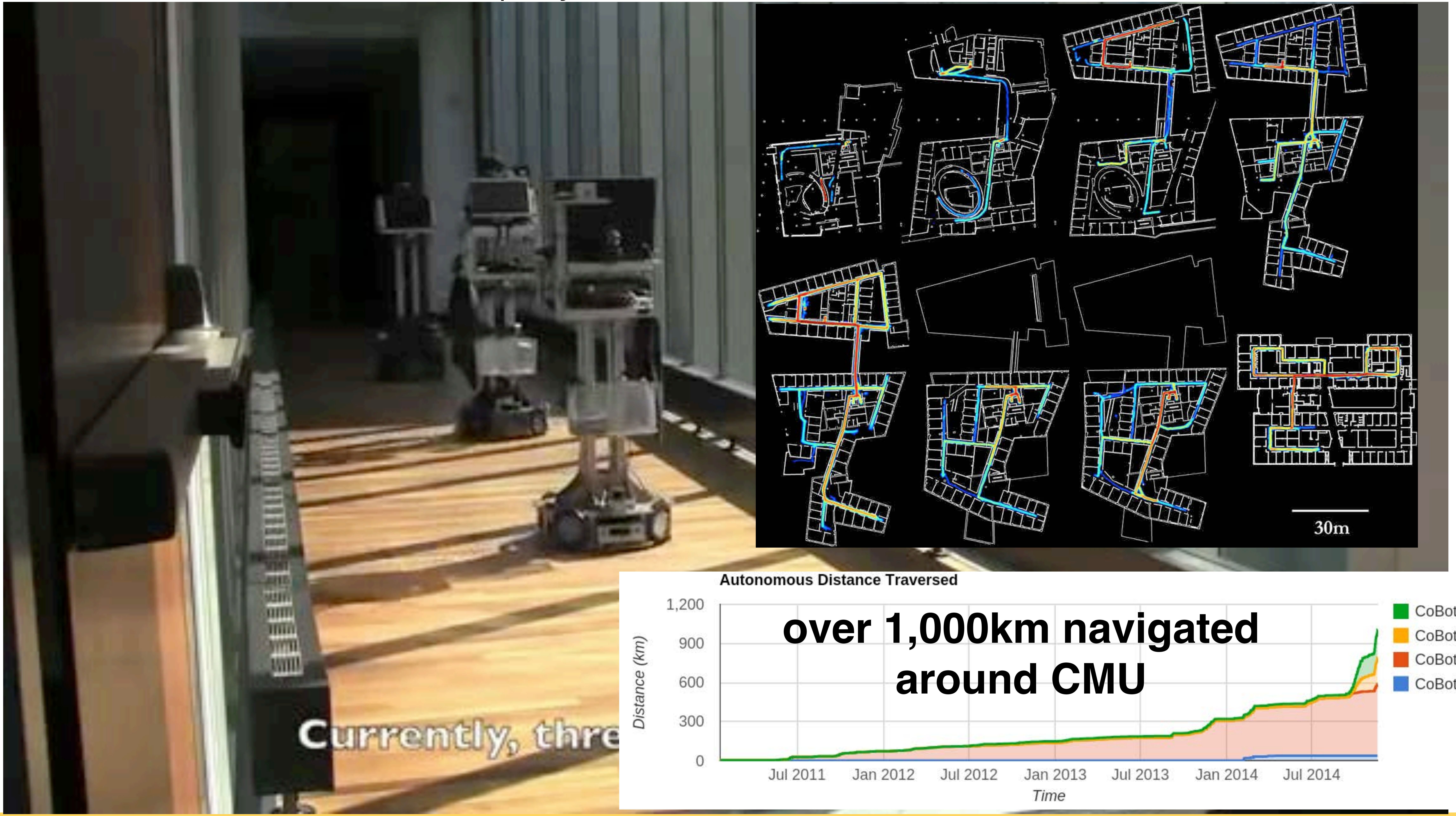
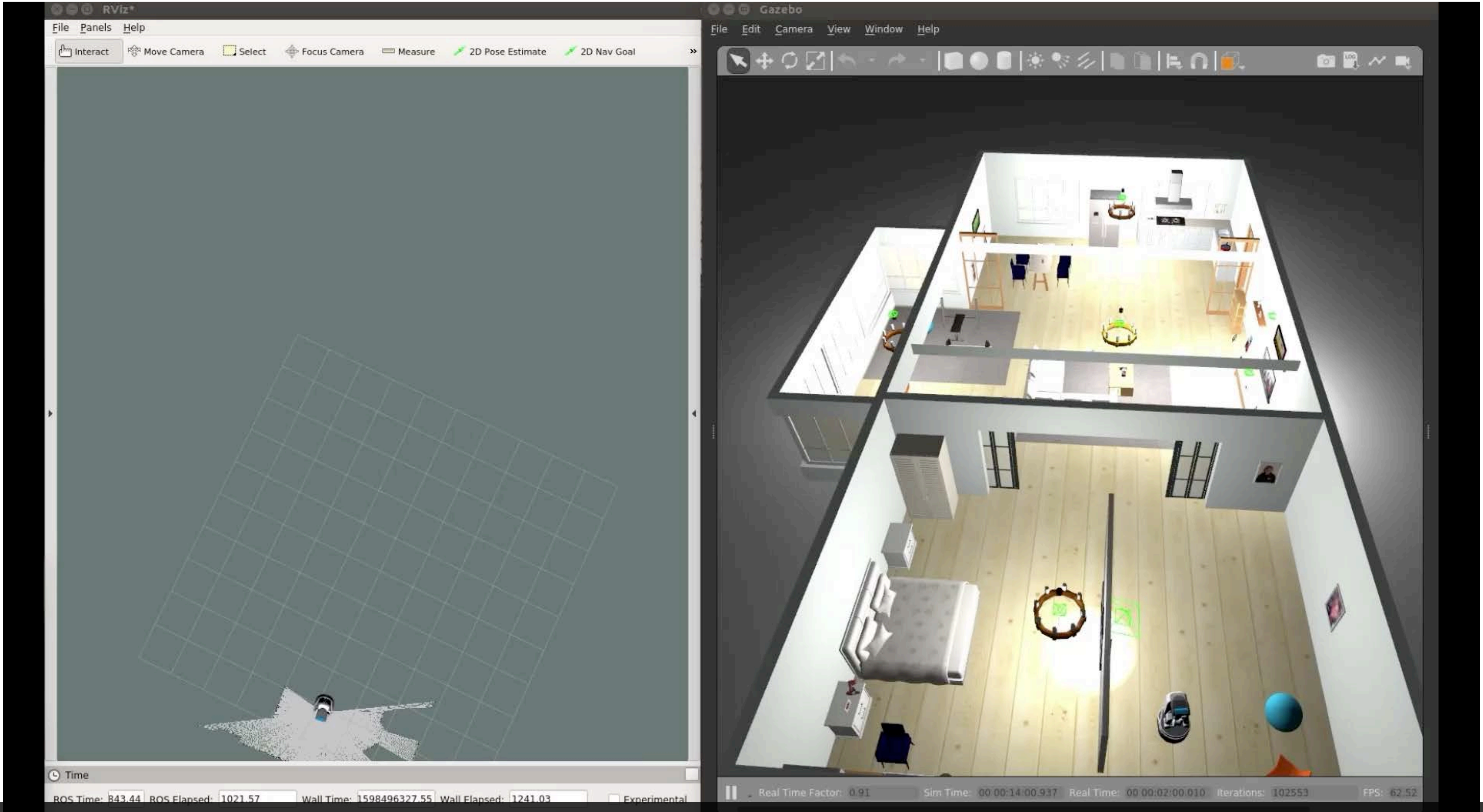
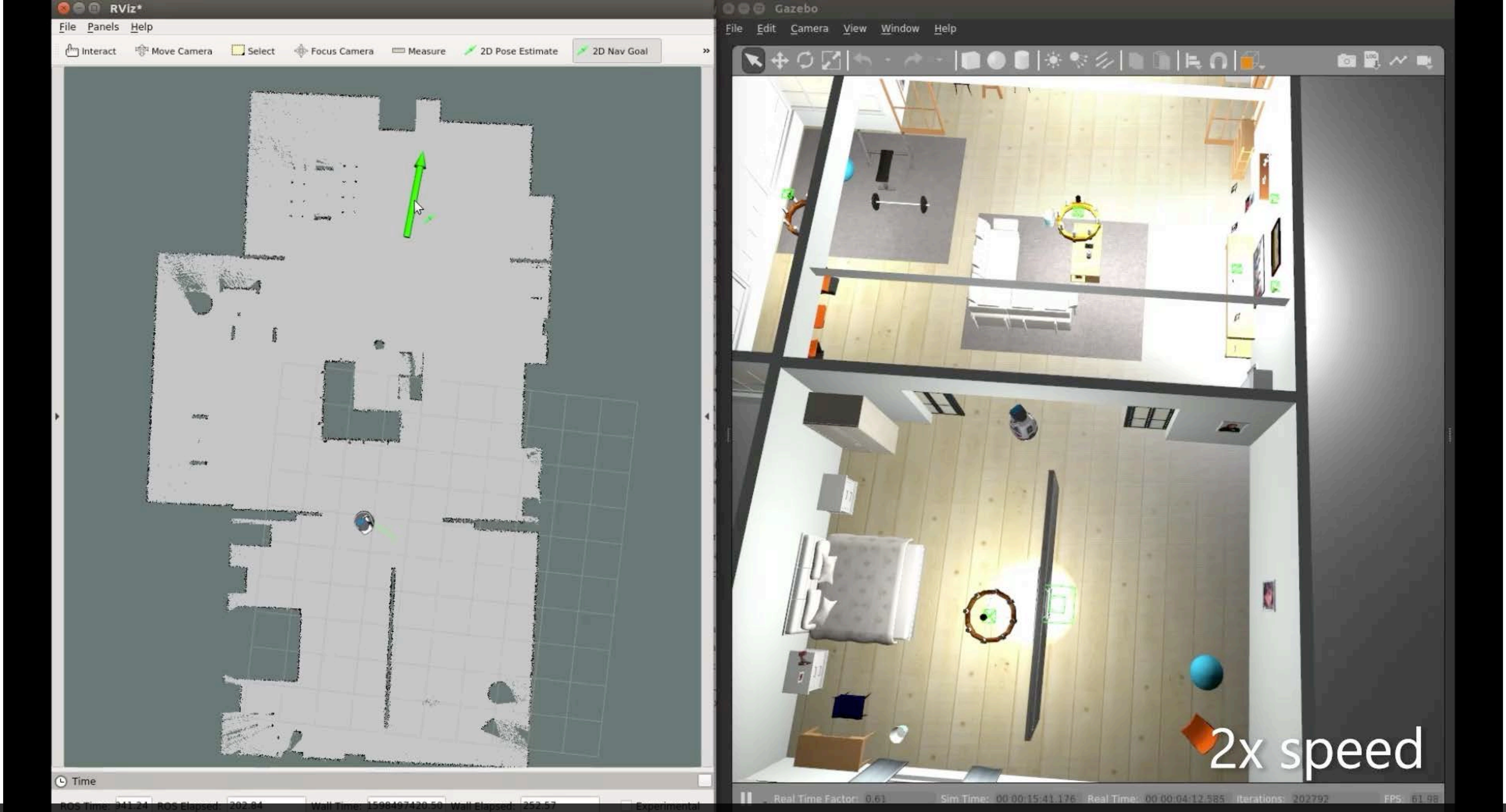


https://www.joydeepb.com/research.html



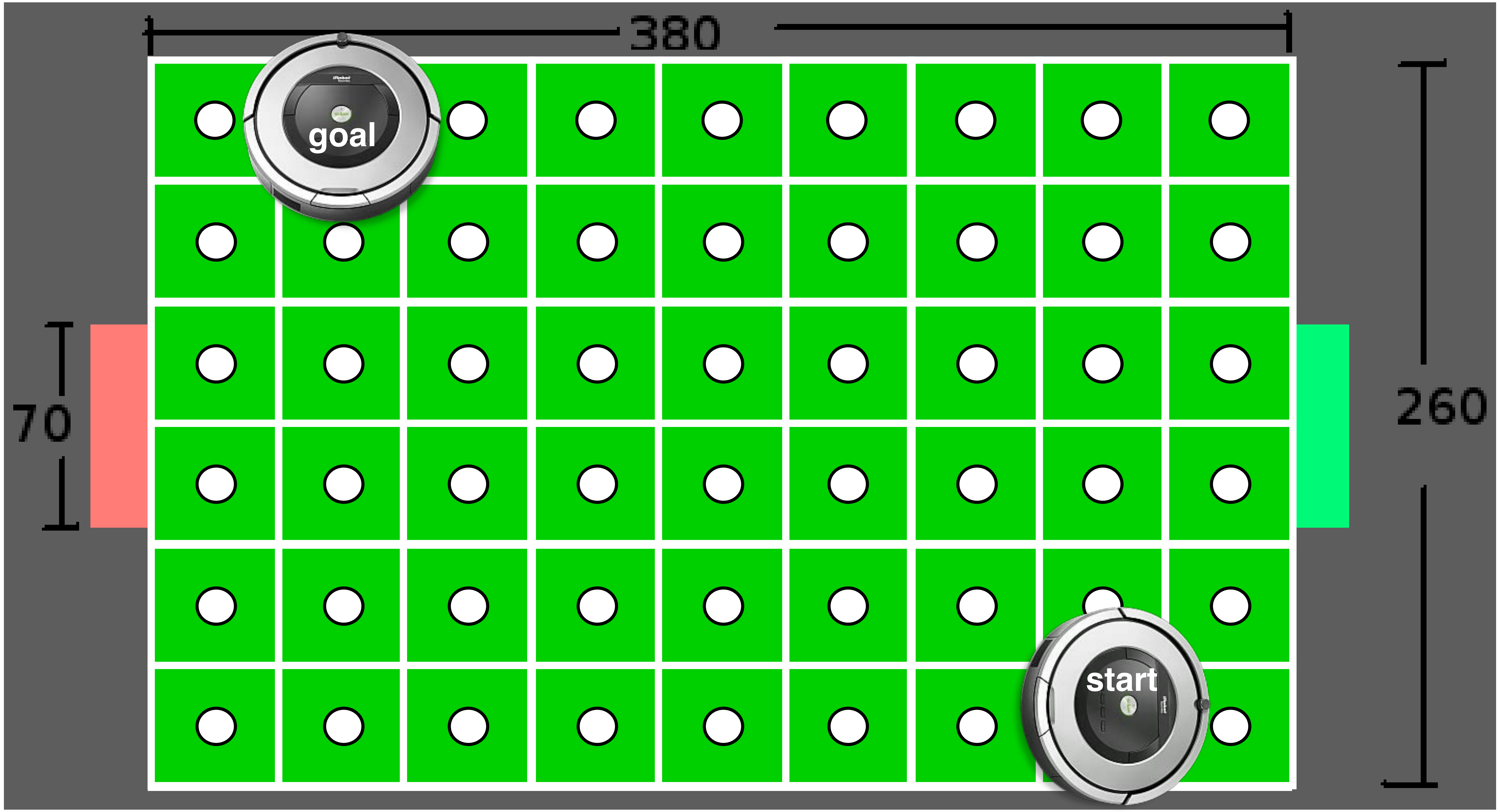
Localization and Mapping - Alphonsus Adu-Bredu - https://youtu.be/wH0QhWgtmuA



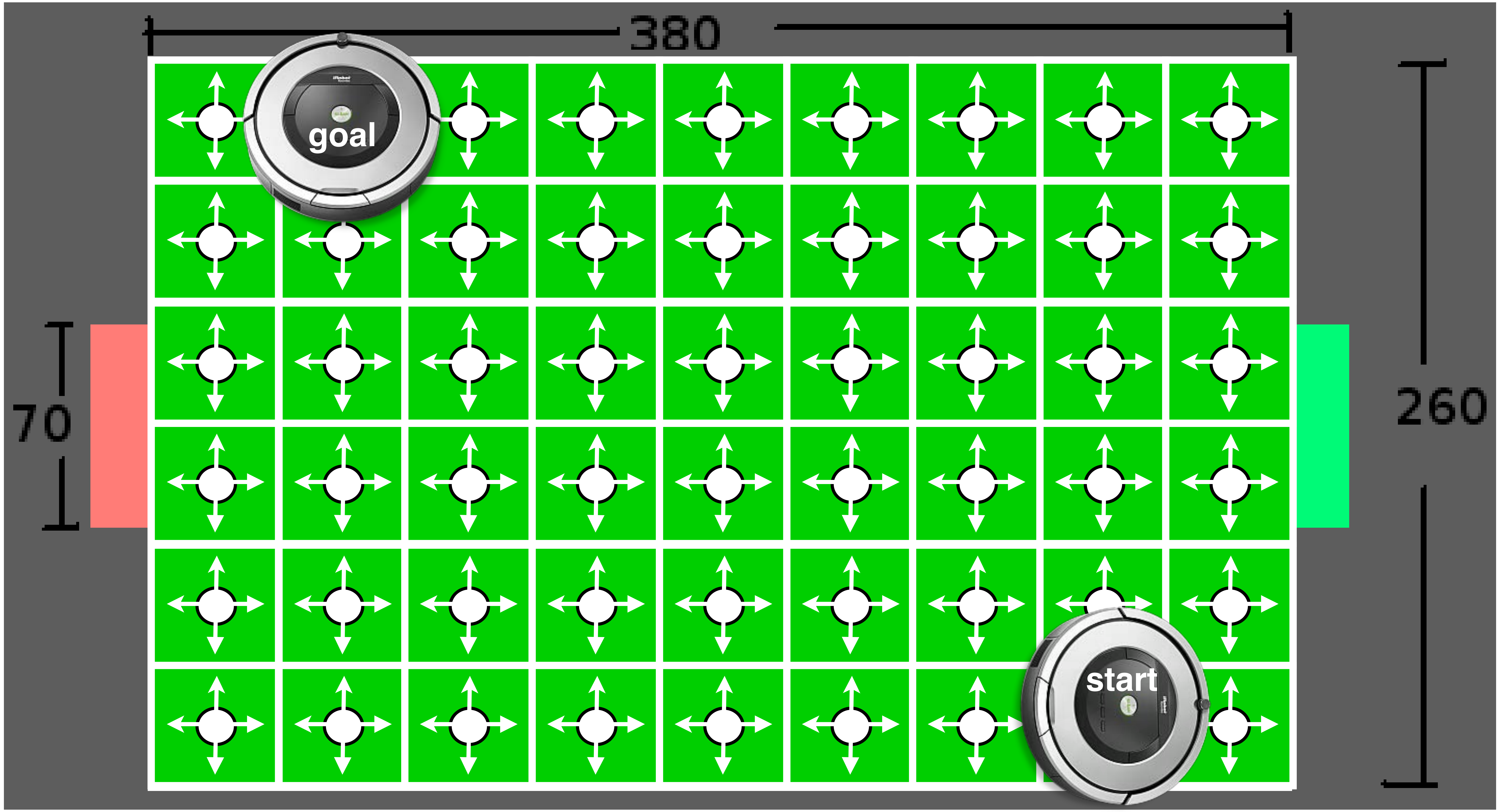
Autonomous Navigation - Alphonsus Adu-Bredu - https://youtu.be/wH0QhWgtmuA

How do we get from A to B?

Consider all possible poses as uniformly distributed array of cells in a graph



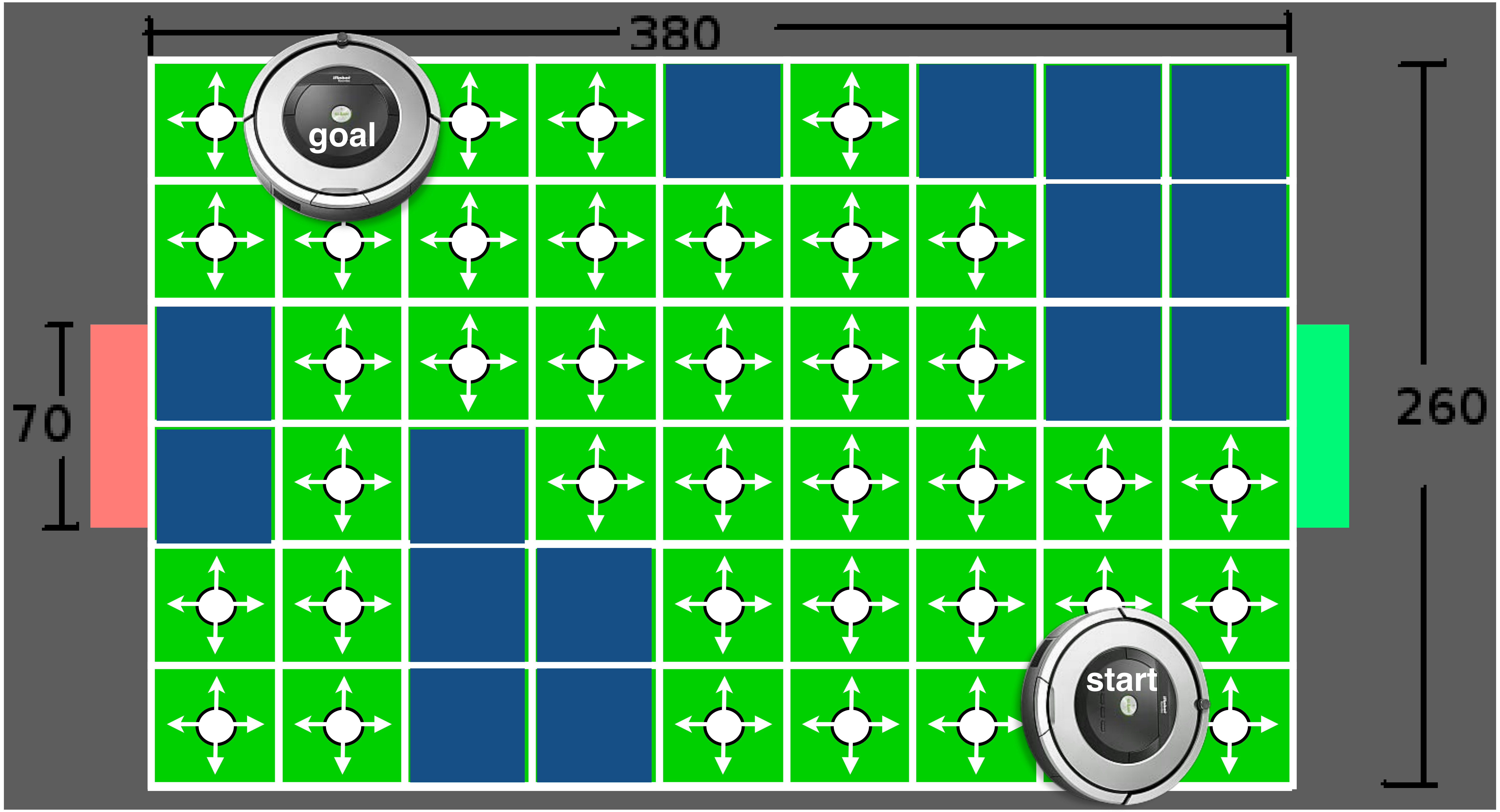
Consider all possible poses as uniformly distributed array of cells in a graph
Edges connect adjacent cells, weighted by distance



Consider all possible poses as uniformly distributed array of cells in a graph

Edges connect adjacent cells, weighted by distance

Cells are invalid where its associated robot pose results in a collision

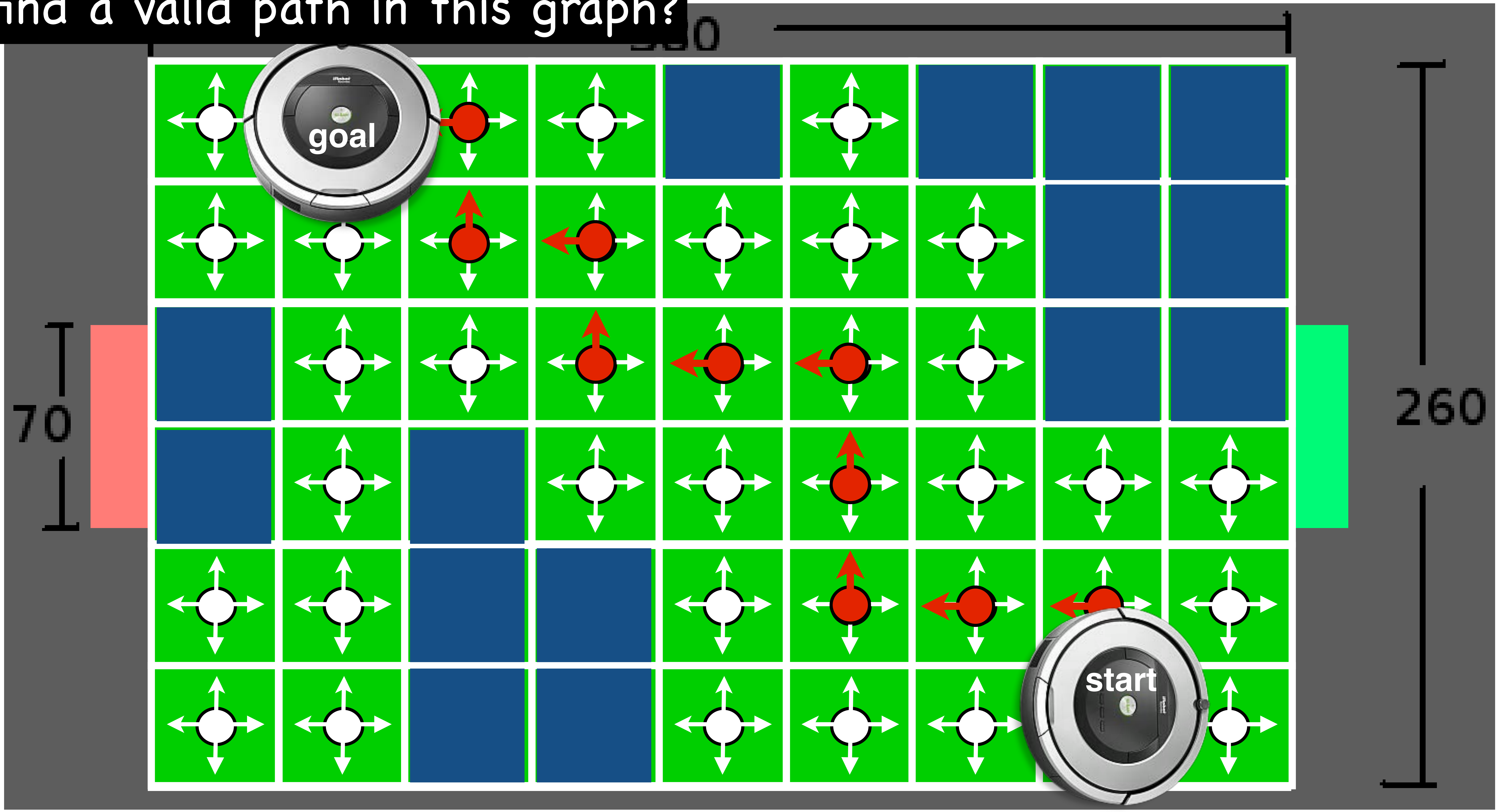


Consider all possible poses as uniformly distributed array of cells in a graph

Edges connect adjacent cells, weighted by distance

Cells are invalid where its associated robot pose results in a collision

How to find a valid path in this graph?



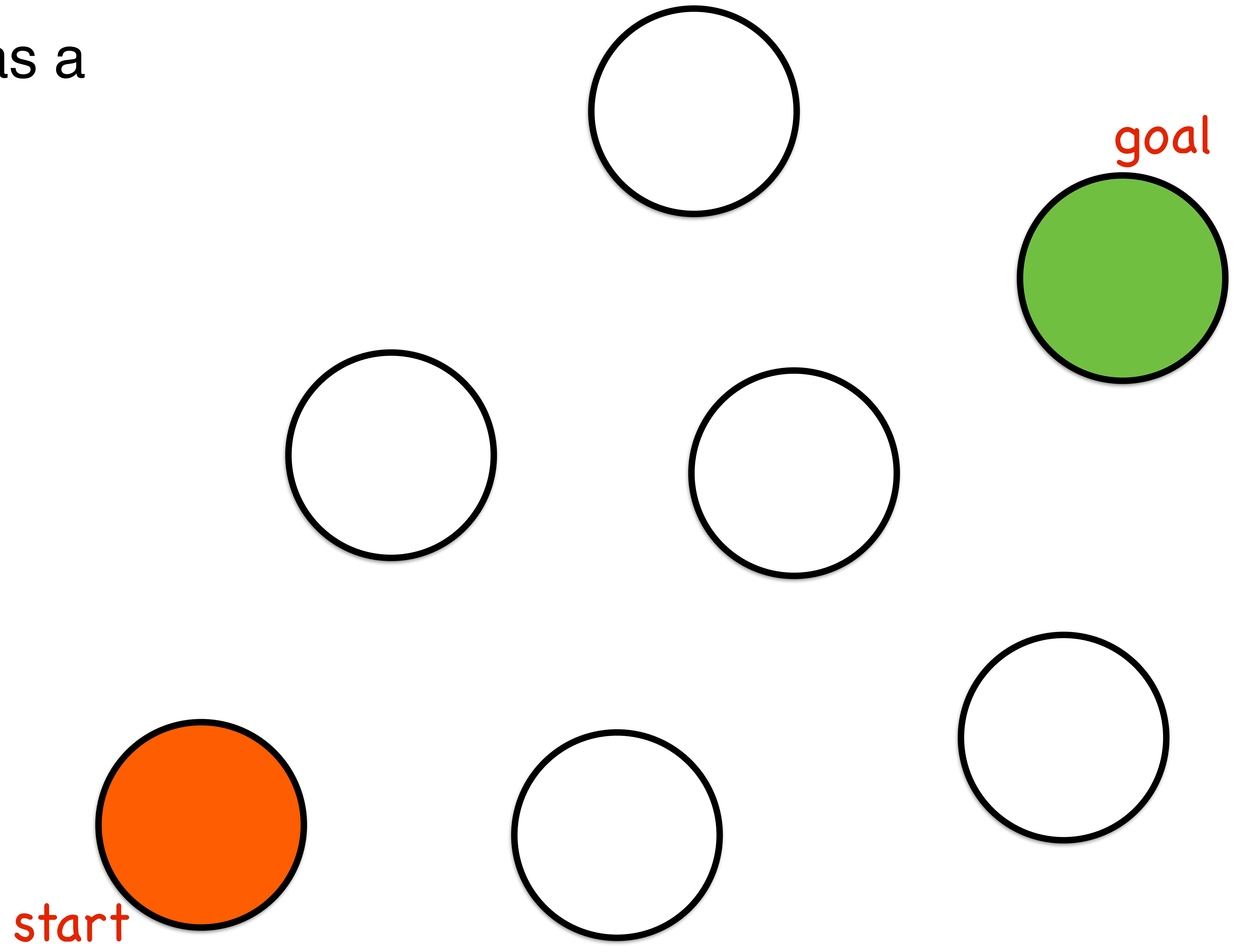
Approaches to *motion* planning

- Bug algorithms: Bug[0-2], Tangent Bug
- **Graph Search (fixed graph)**
  - **Depth-first, Breadth-first, Dijkstra, A-star,** Greedy best-first
- Sampling-based Search (build graph):
  - Probabilistic Road Maps, Rapidly-exploring Random Trees
- Optimization (local search):
  - Gradient descent, potential fields, Wavefront

Consider a simple search graph

Consider a simple search graph

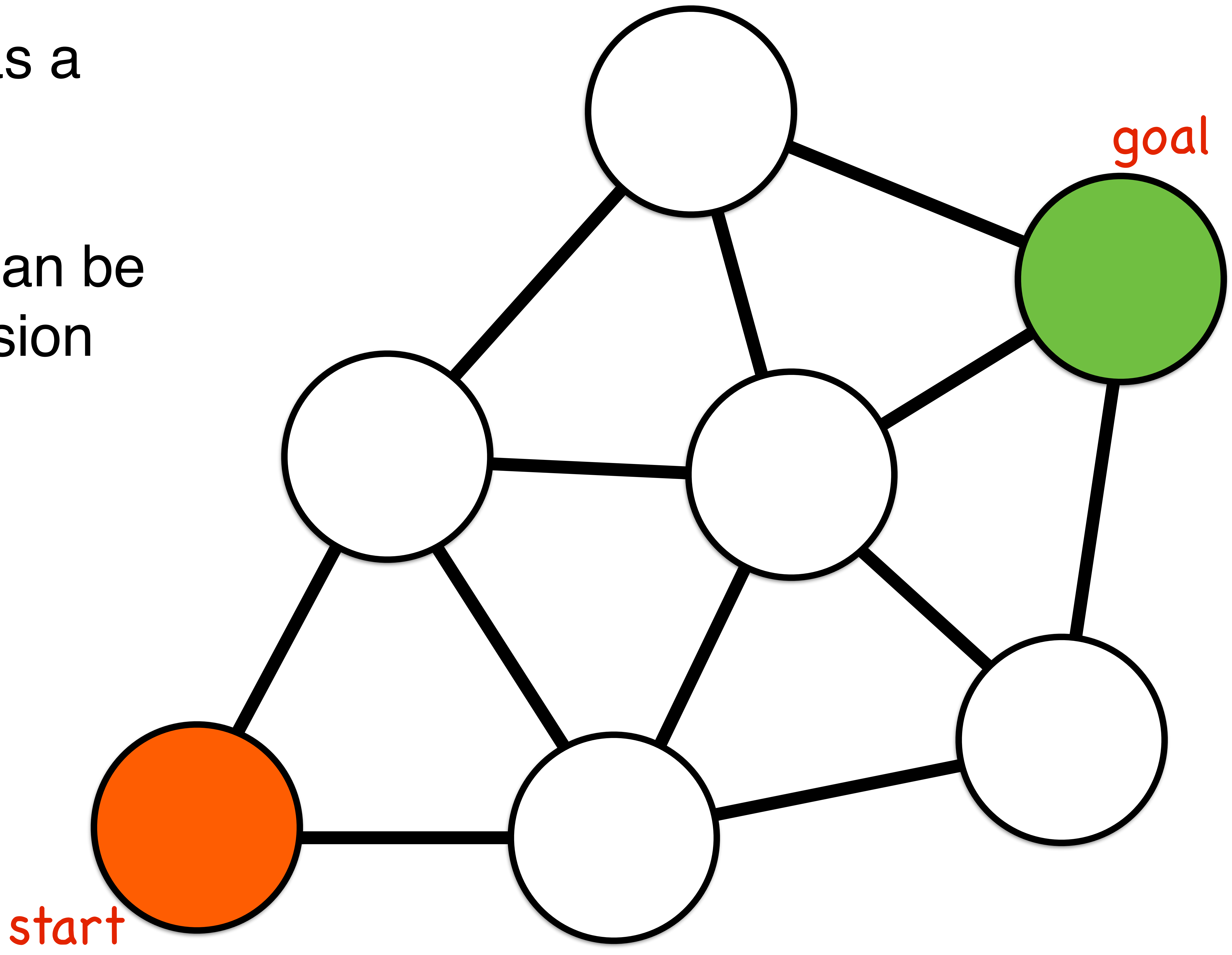
Consider each possible robot pose as a node V_i in a graph $G(V,E)$



Consider a simple search graph

Consider each possible robot pose as a node V_i in a graph $G(V,E)$

Graph edges E connect poses that can be reliably moved between without collision

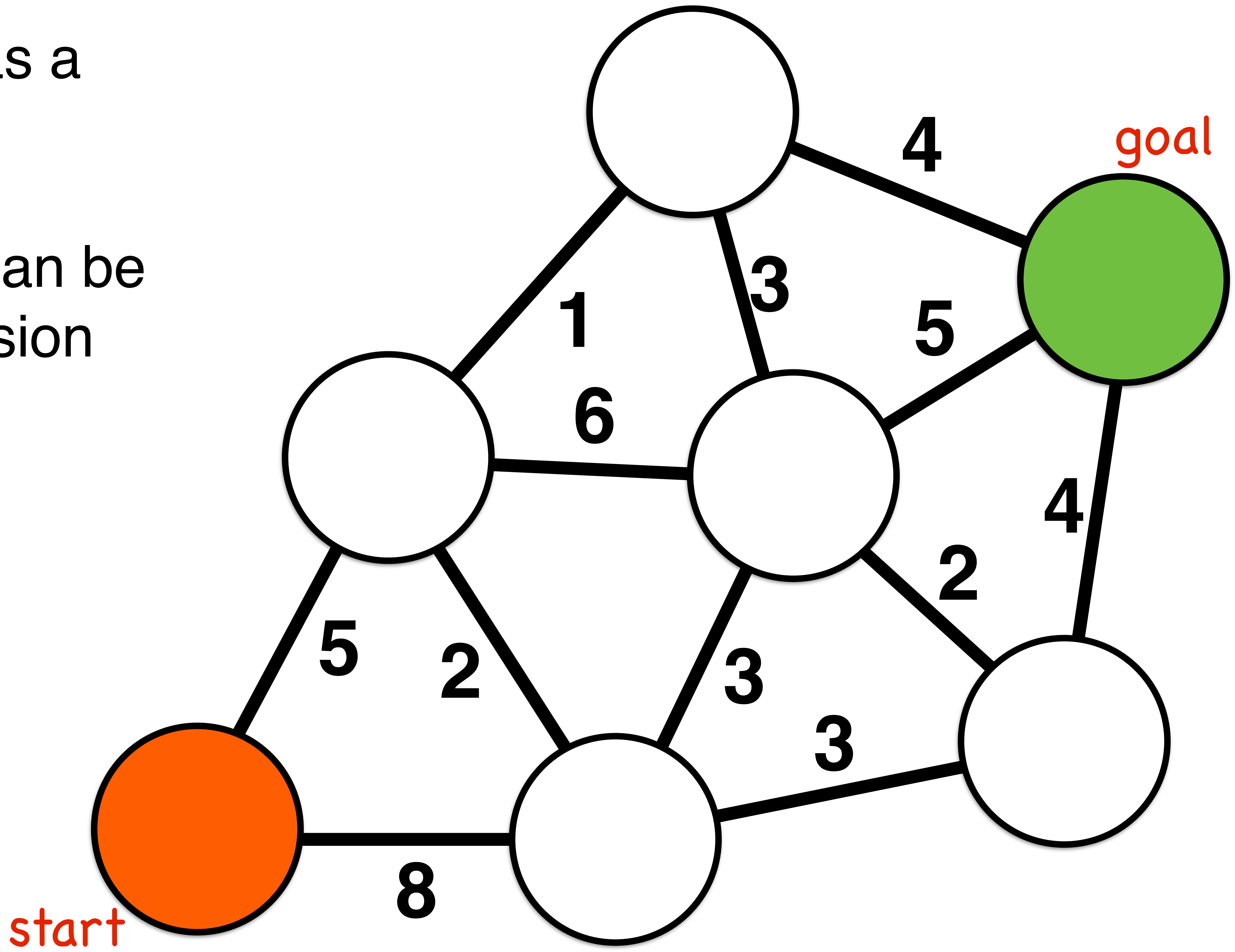


Consider a simple search graph

Consider each possible robot pose as a node V_i in a graph $G(V,E)$

Graph edges E connect poses that can be reliably moved between without collision

Edges have a cost for traversal



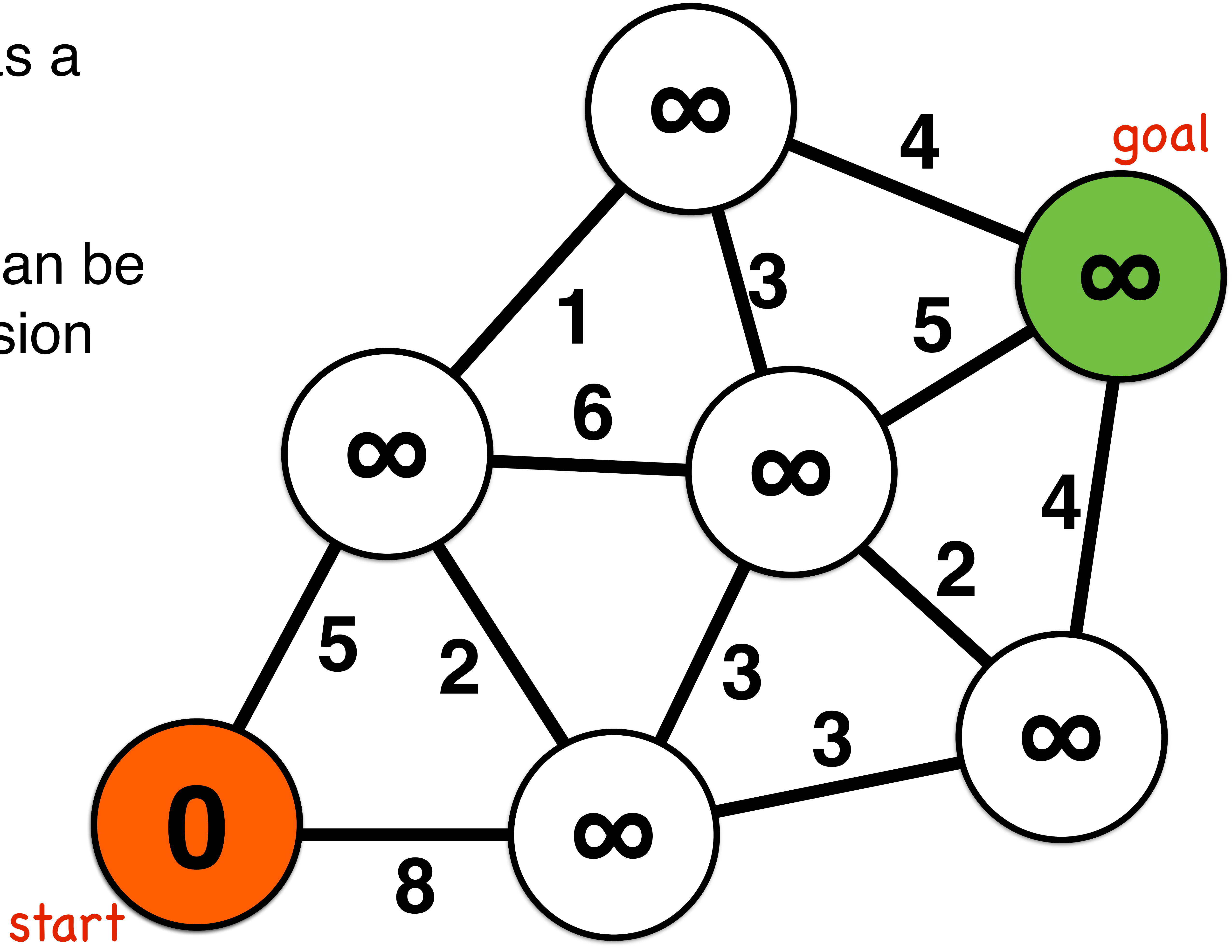
Consider a simple search graph

Consider each possible robot pose as a node V_i in a graph $G(V,E)$

Graph edges E connect poses that can be reliably moved between without collision

Edges have a cost for traversal

Each node maintains the **distance** traveled from start as a scalar cost



Consider a simple search graph

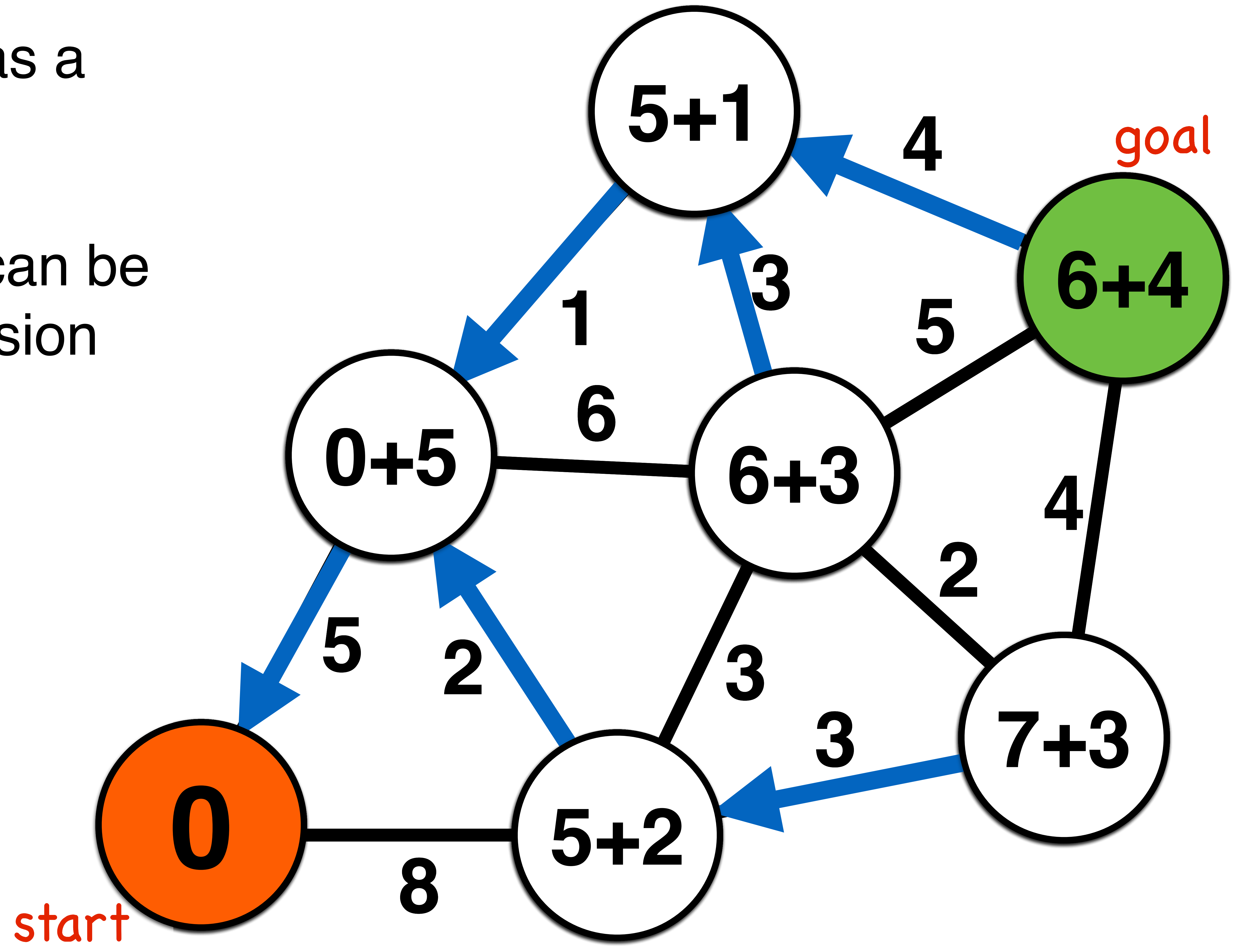
Consider each possible robot pose as a node V_i in a graph *G(V,E)*

Graph edges *E* connect poses that can be reliably moved between without collision

Edges have a cost for traversal

Each node maintains the **distance** traveled from start as a scalar cost

Each node has a **parent** node that specifies its route to the start node

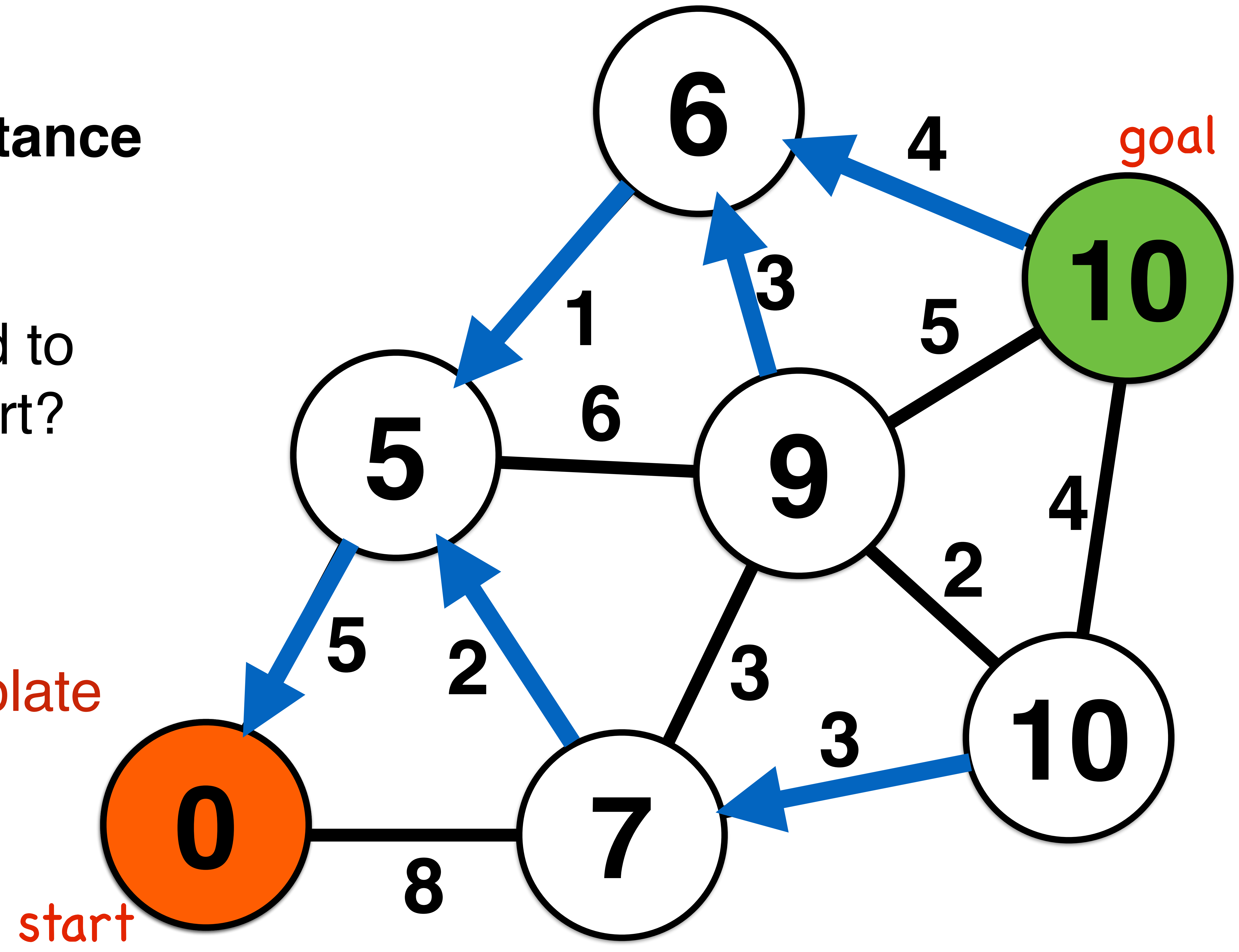


Path Planning as Graph Search

Which route is best to optimize **distance** traveled from start?

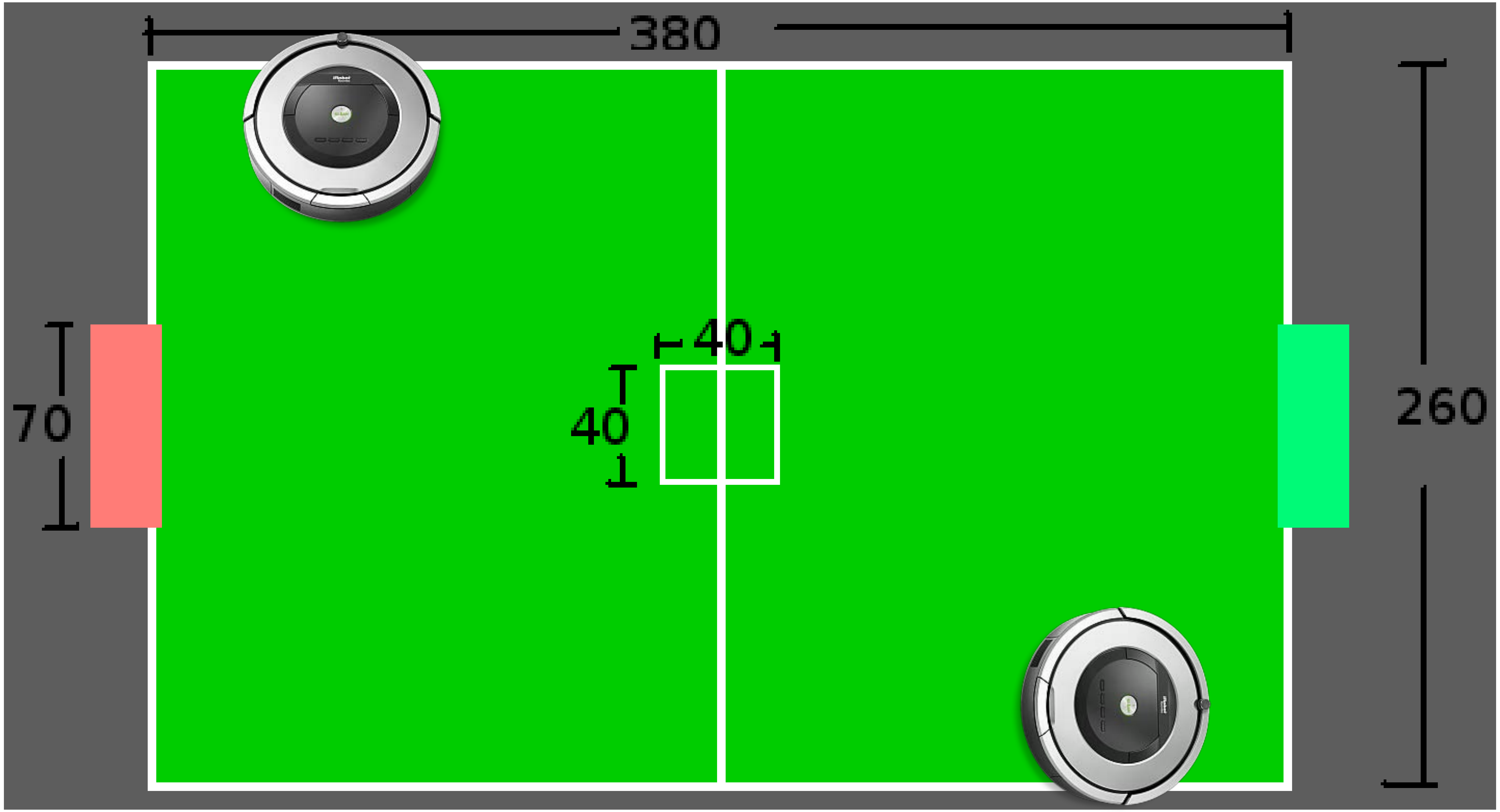
Which **parent** node should be used to specify route between goal and start?

We will use a single algorithm template for our graph search computation



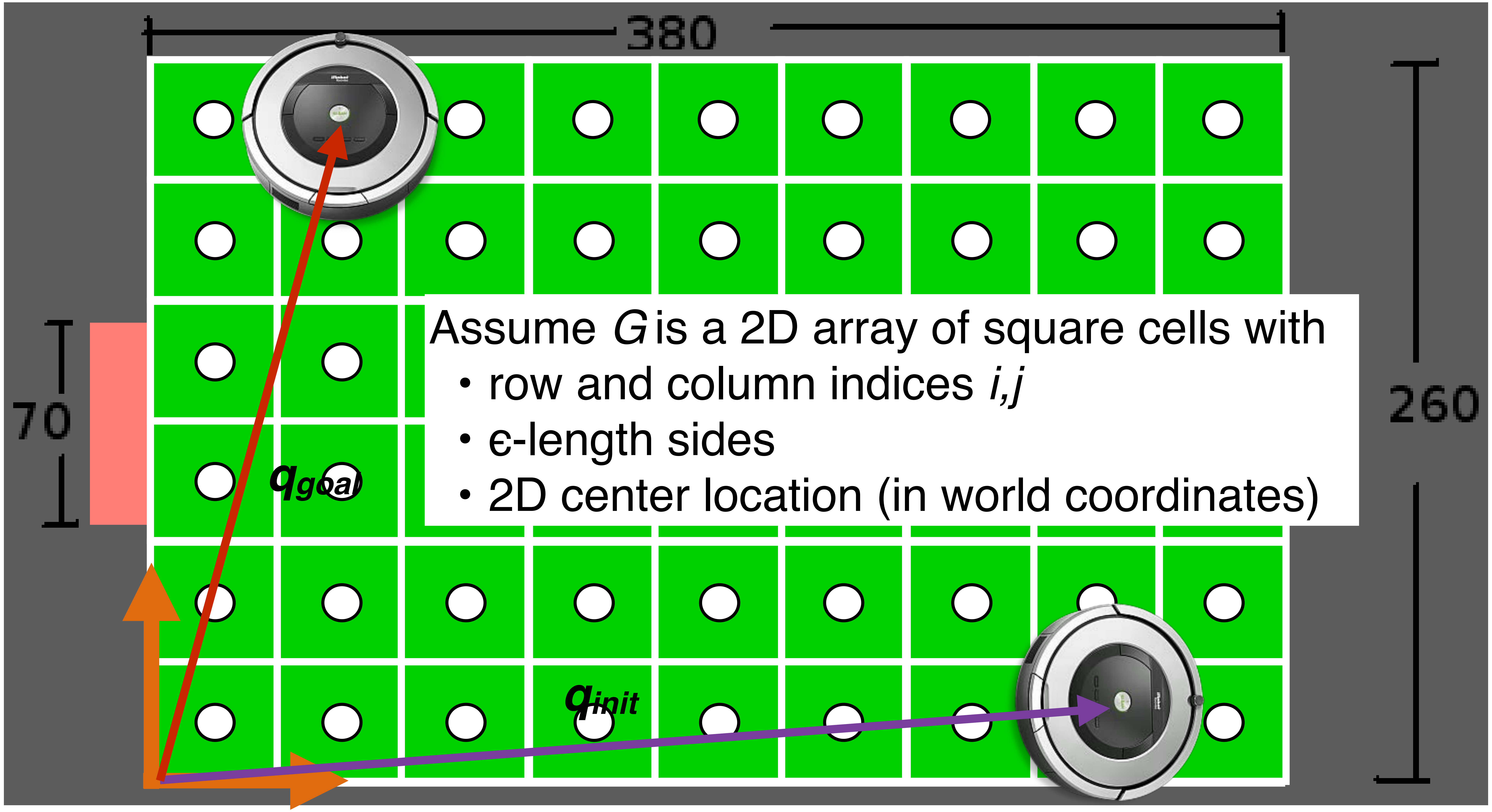
Depth-first search intuition and walkthrough

Depth-first search



Depth-first search

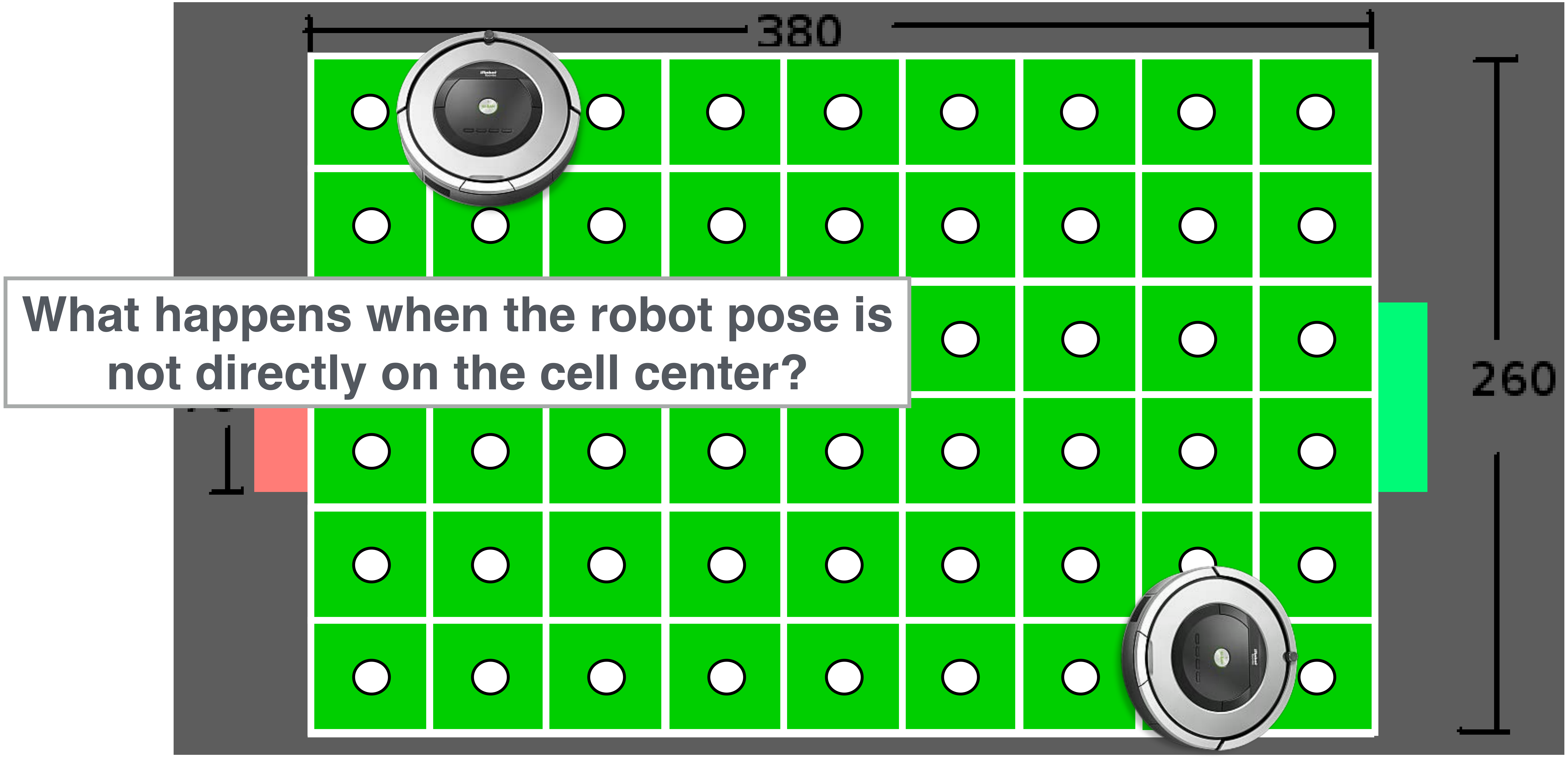
Depth-first search



Assume *G* is a 2D array of square cells with

- row and column indices *i,j*
- є-length sides
- 2D center location (in world coordinates)

Depth-first search



Graph Accessibility

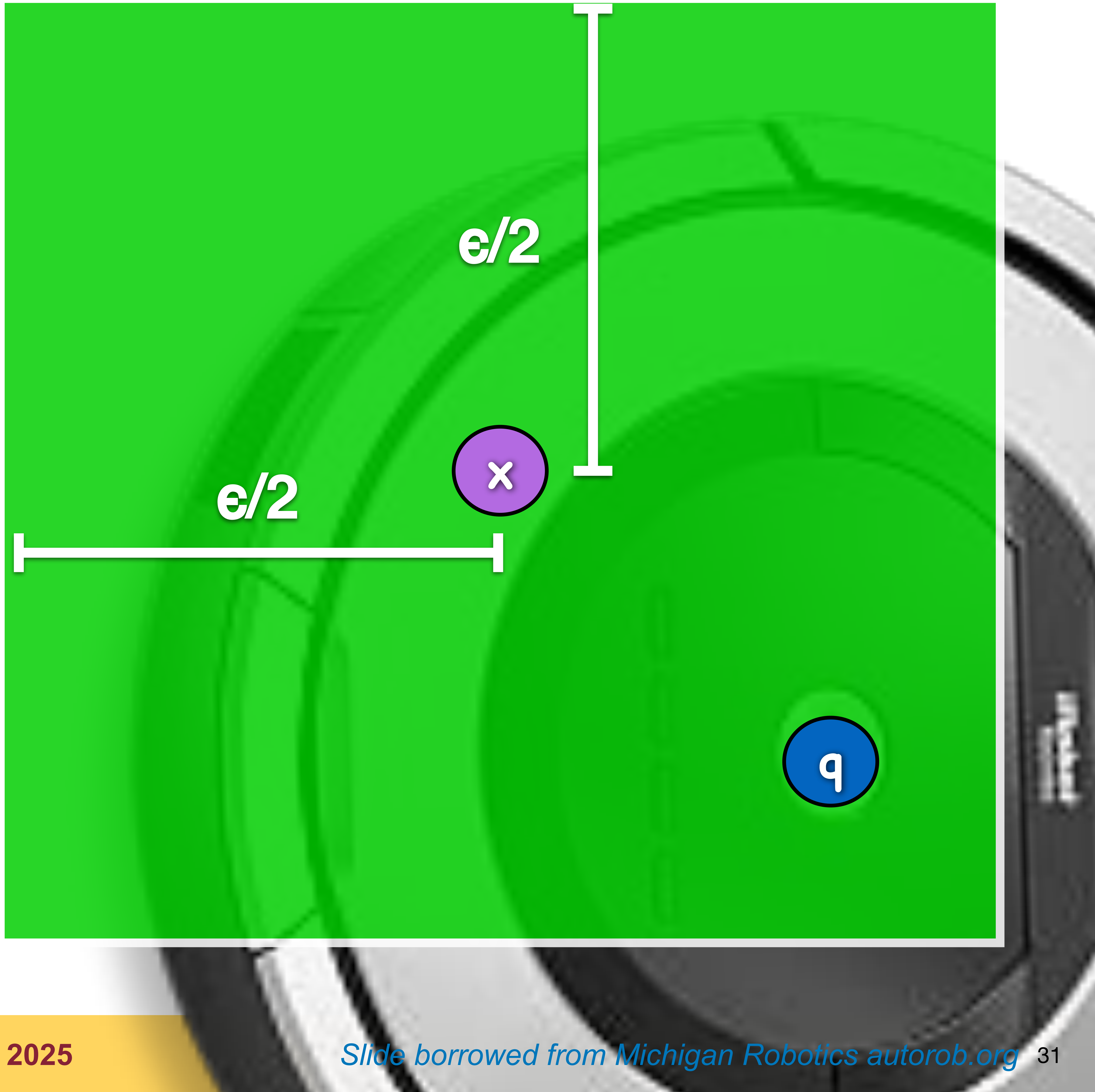
What happens when the robot pose is not directly on the cell center?

Graph Accessibility

A graph node $G_{i,j}$ represents a region of space contained by its cell

Start node: the robot accesses graph G at the cell that contains location q_{init}

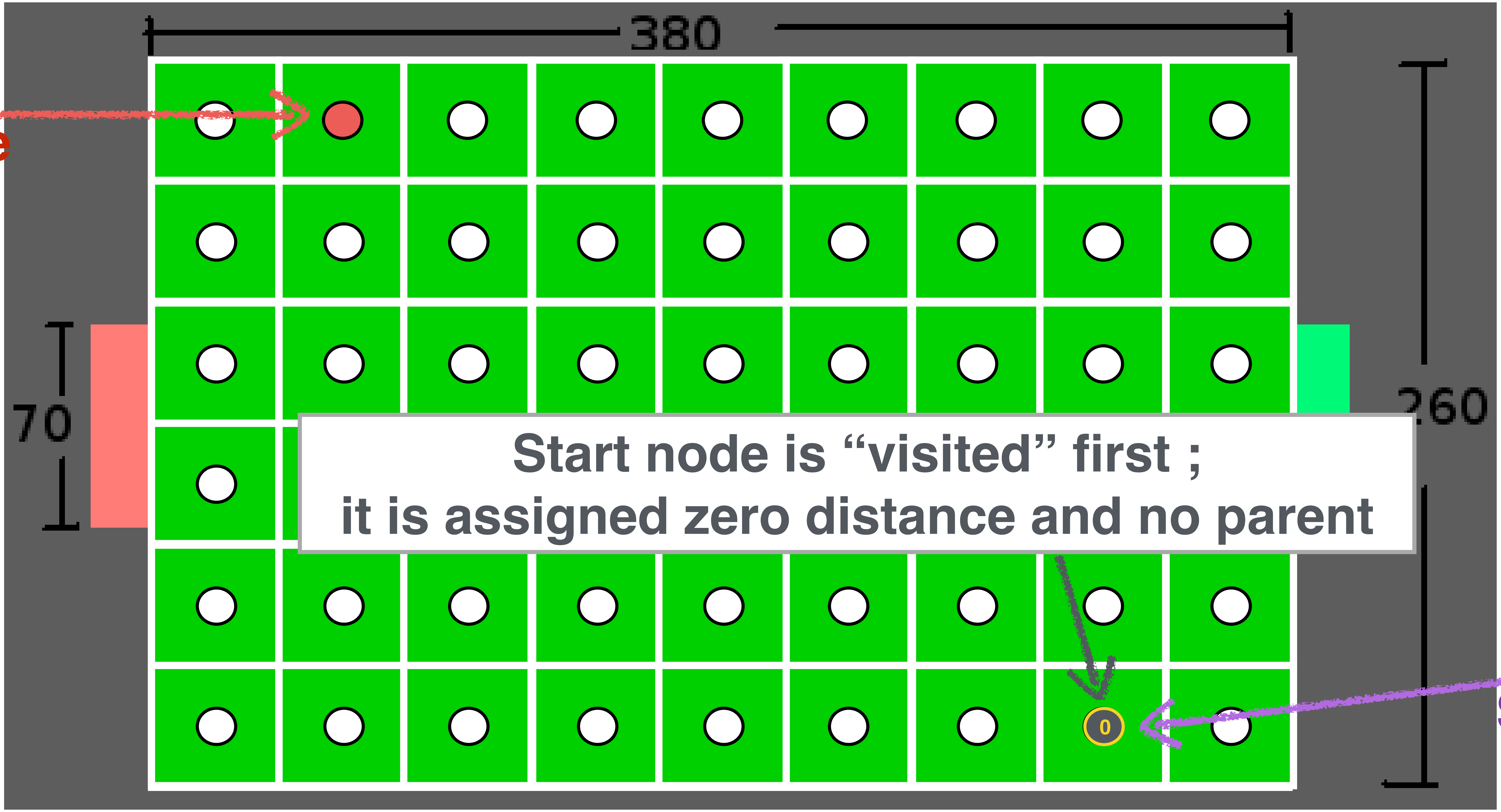
Goal node: the robot departs graph G at the cell that contains location q_{goal}



Depth-first search

Depth-first search

Goal node



Start node is "visited" first ;
it is assigned zero distance and no parent

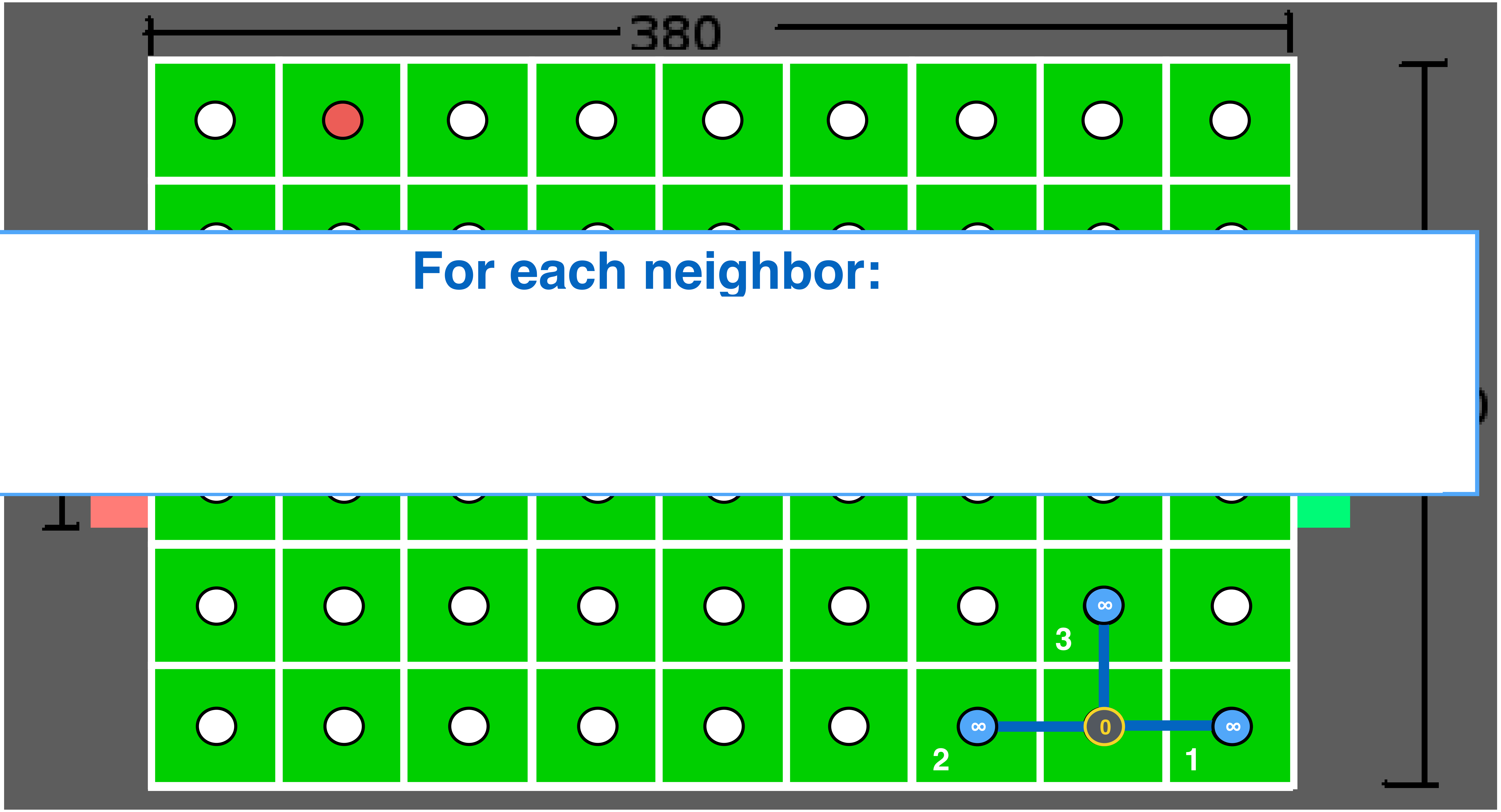
Start node

Depth-first search

Slide borrowed from Michigan Robotics autorob.org

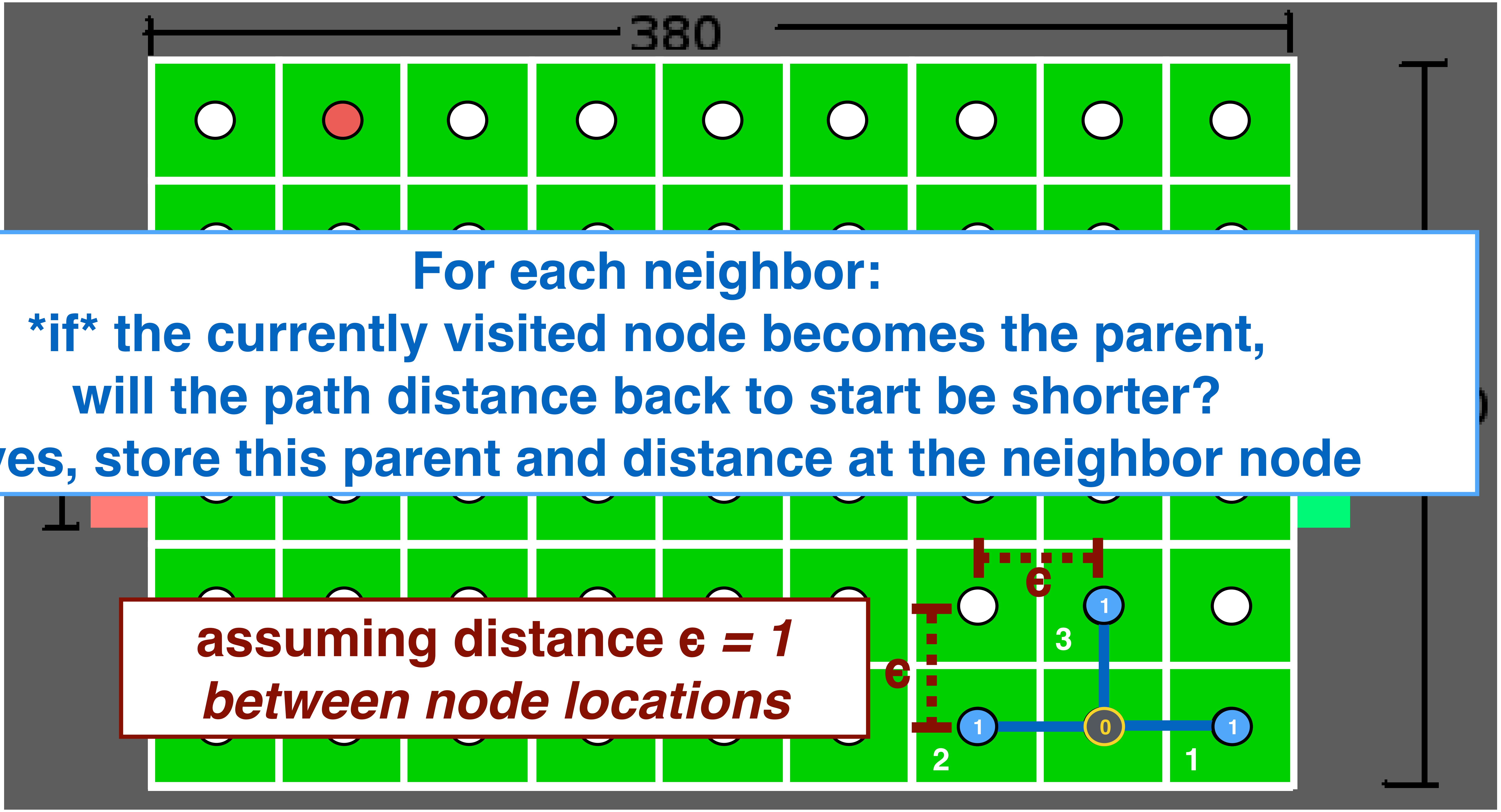
Depth-first search

Depth-first search



For each neighbor:

Depth-first search

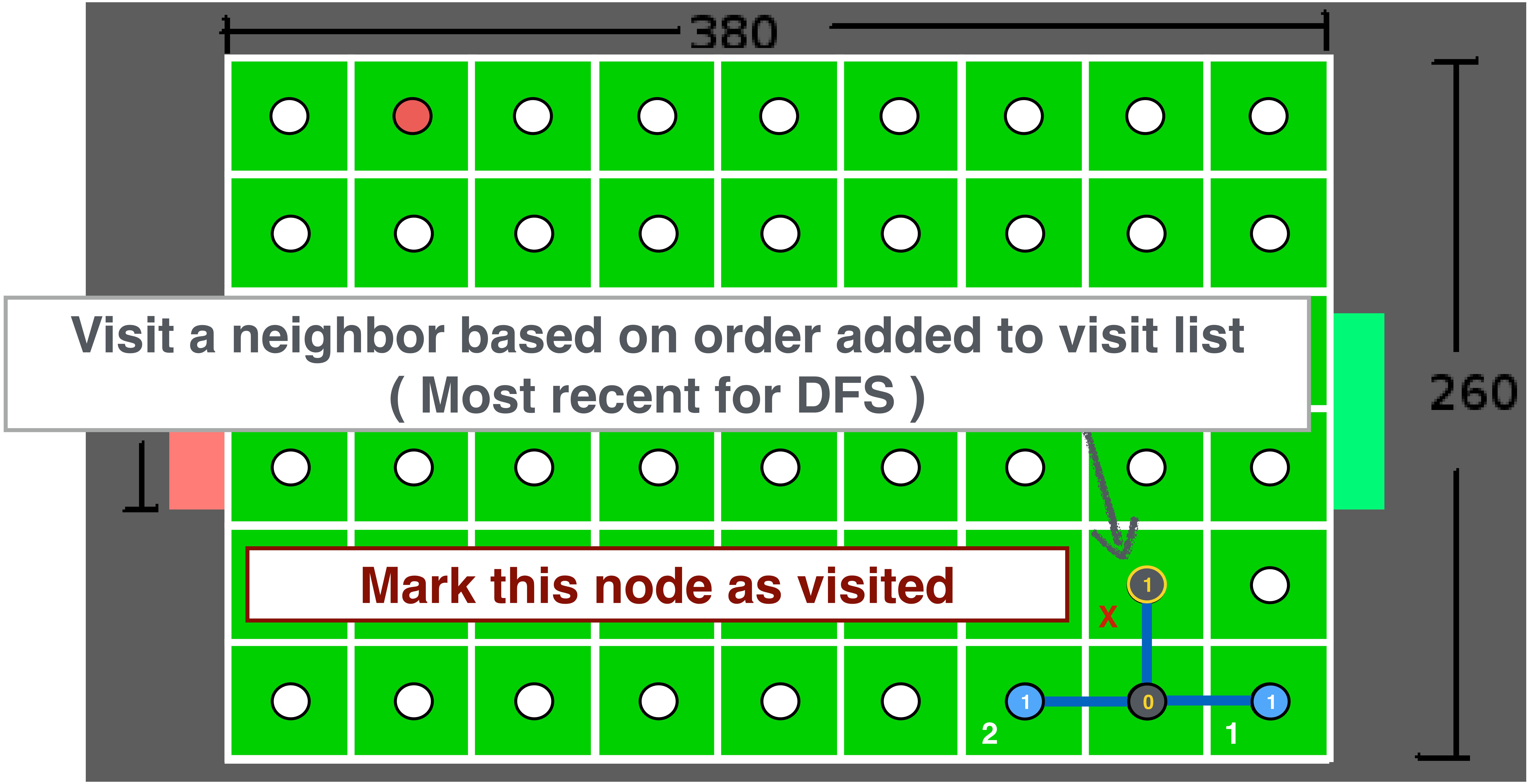


For each neighbor:
***if* the currently visited node becomes the parent,**
will the path distance back to start be shorter?
if yes, store this parent and distance at the neighbor node

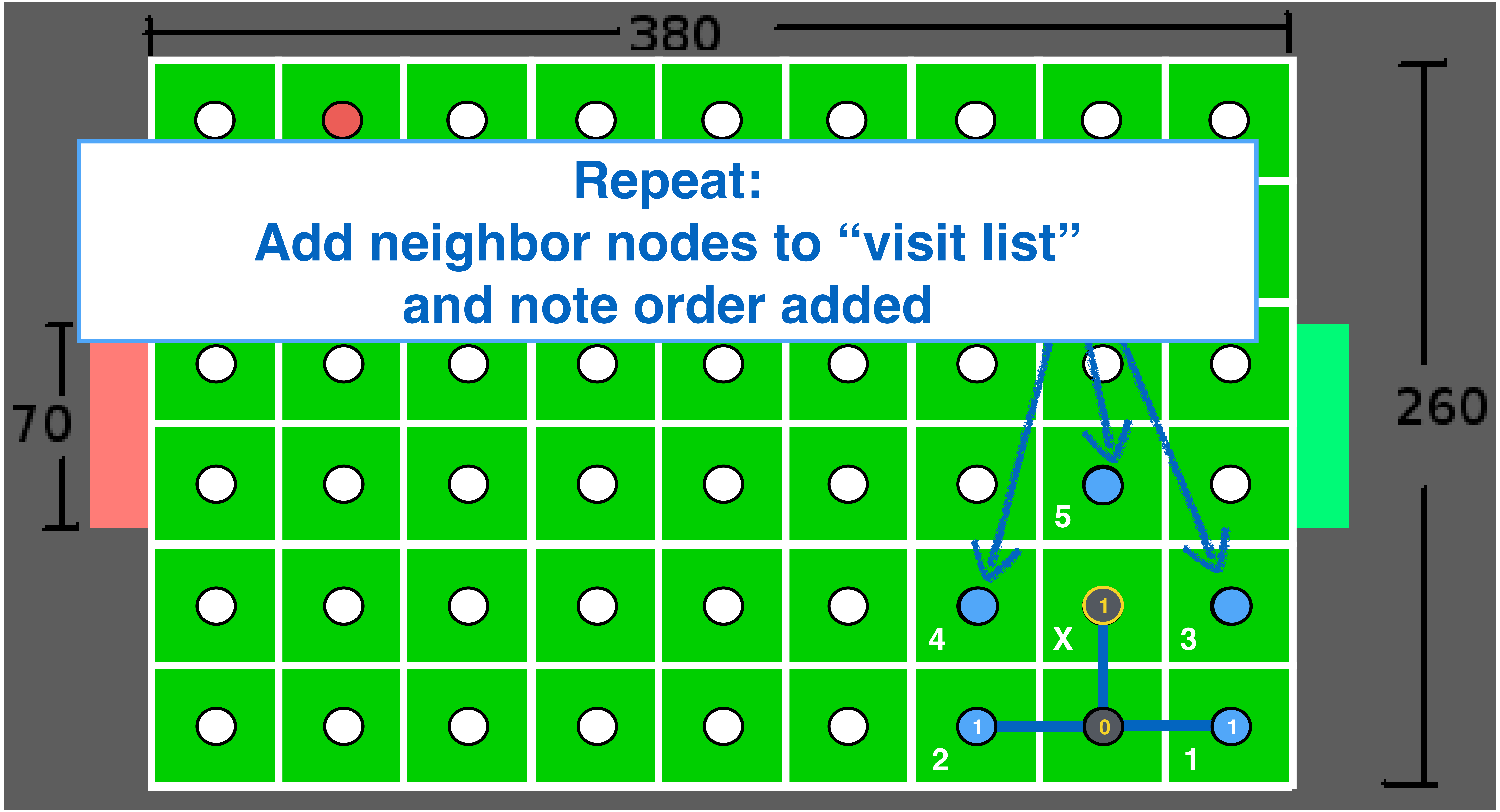
assuming distance ϵ = *1*
between node locations

Depth-first search

Depth-first search



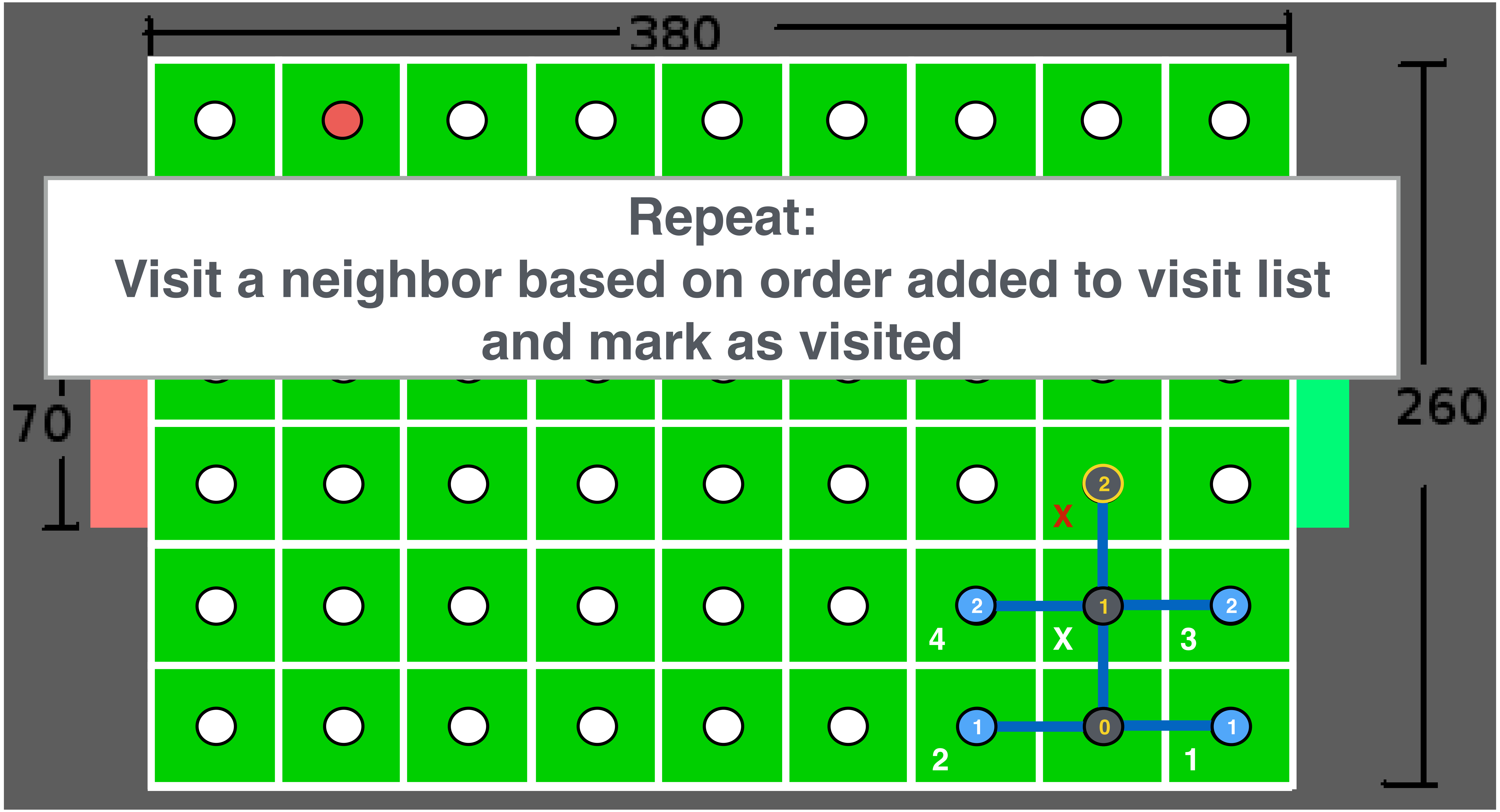
Depth-first search



Depth-first search

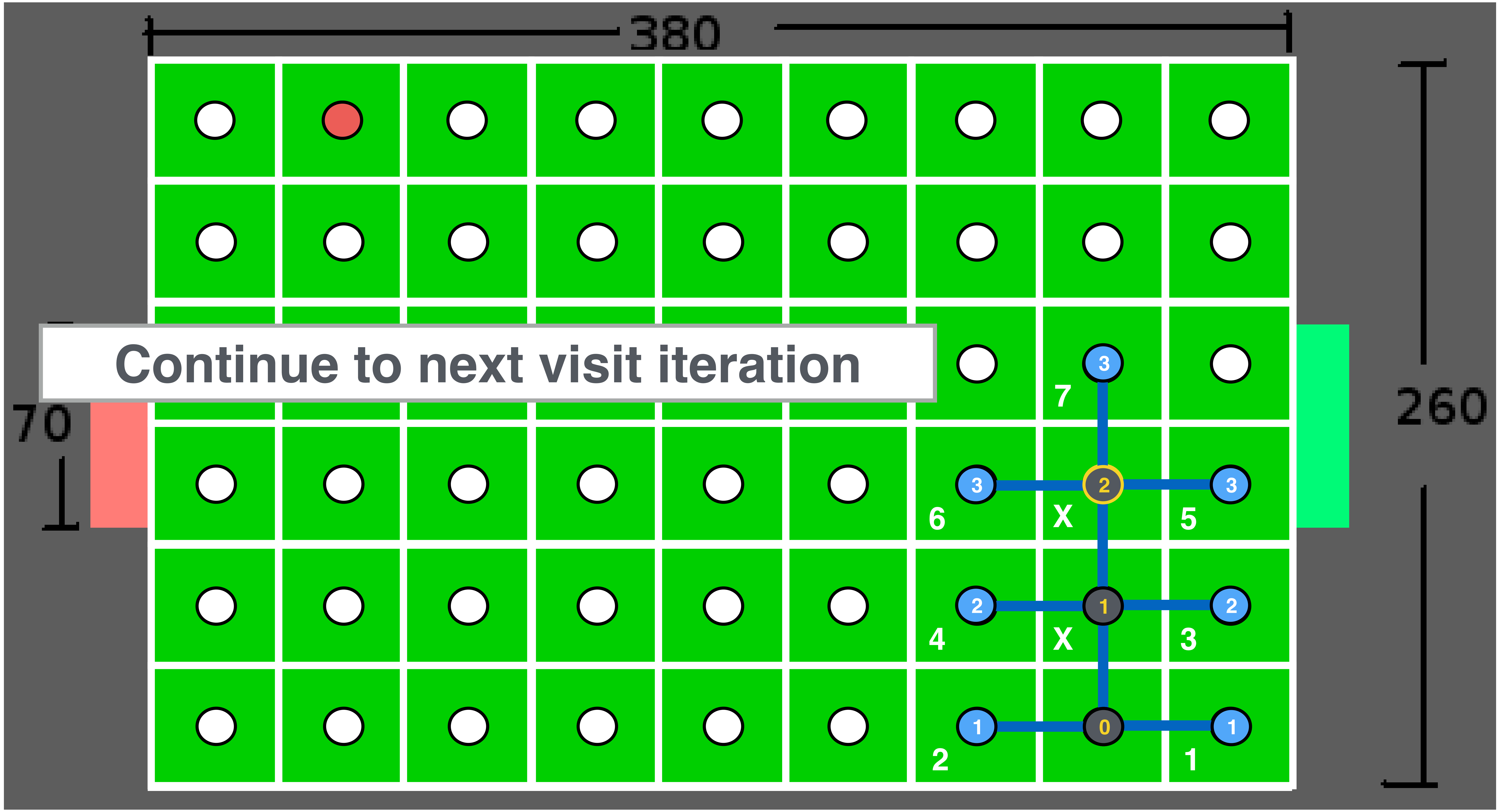
Repeat:
For each neighbor:
choose parent node that minimizes path distance back to start
***AND* store this distance (є + ①) at the neighbor node**

Depth-first search



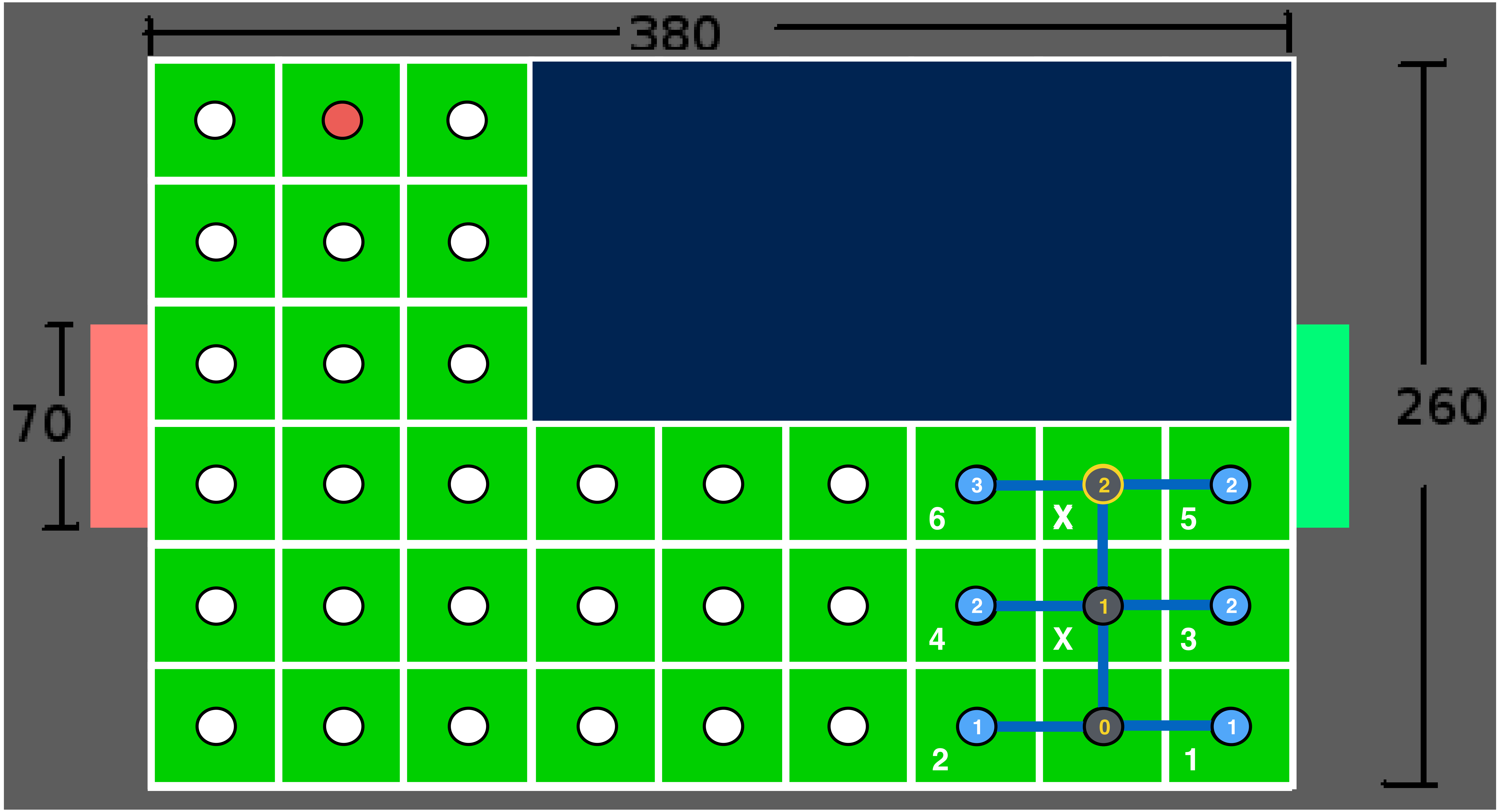
Repeat:
Visit a neighbor based on order added to visit list and mark as visited

Depth-first search



Depth-first search

Depth-first search



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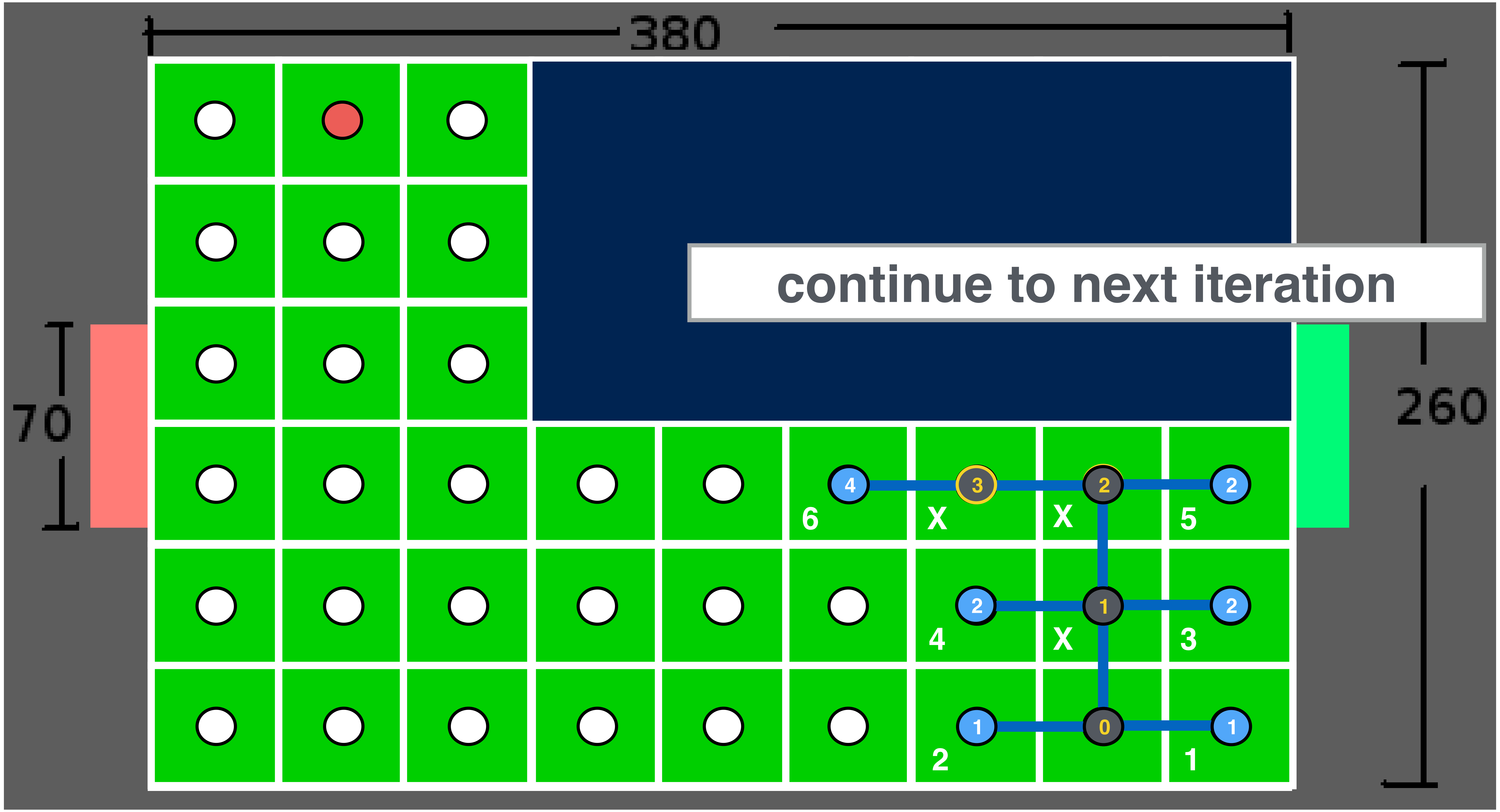

Depth-first search

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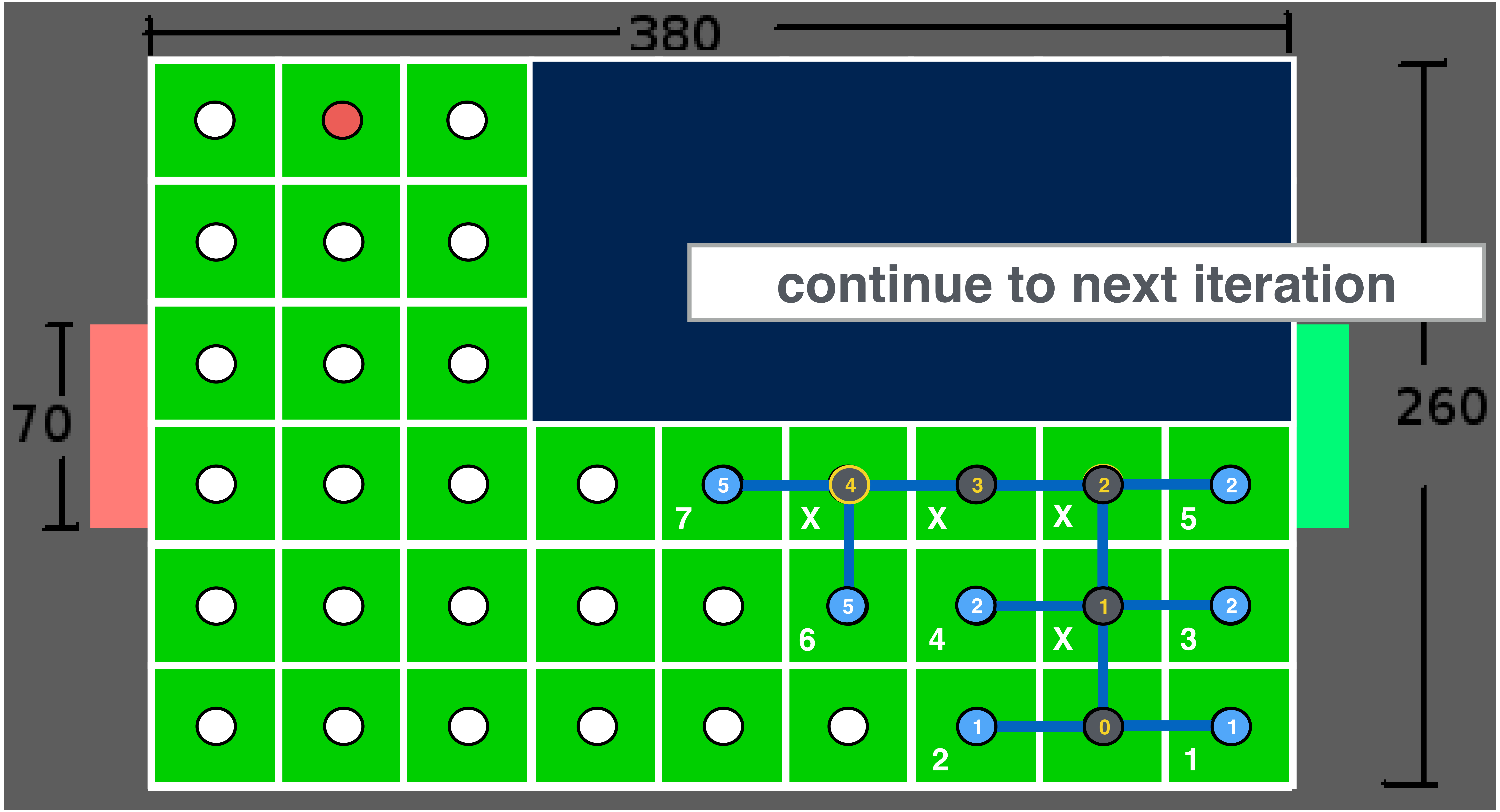
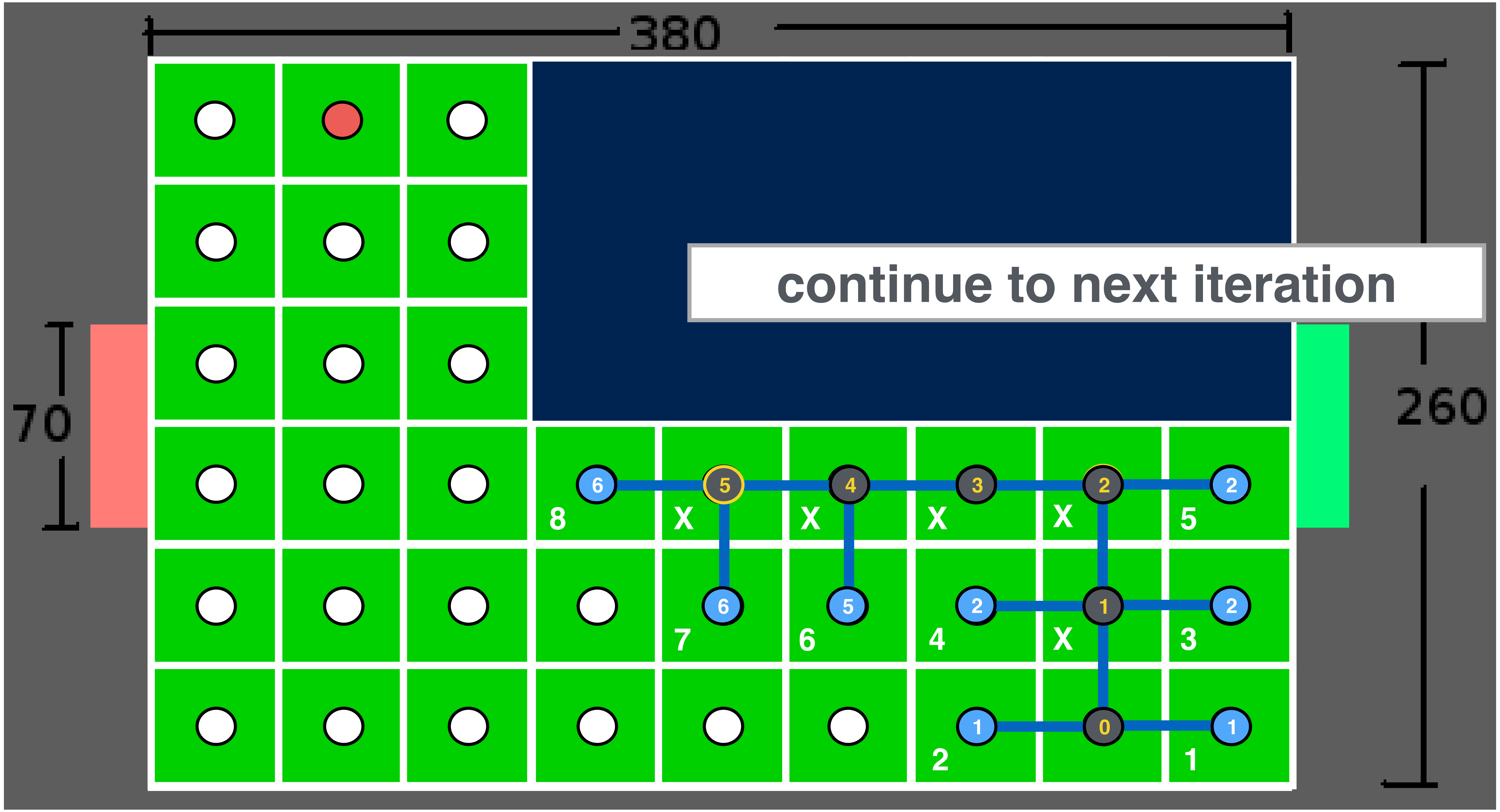
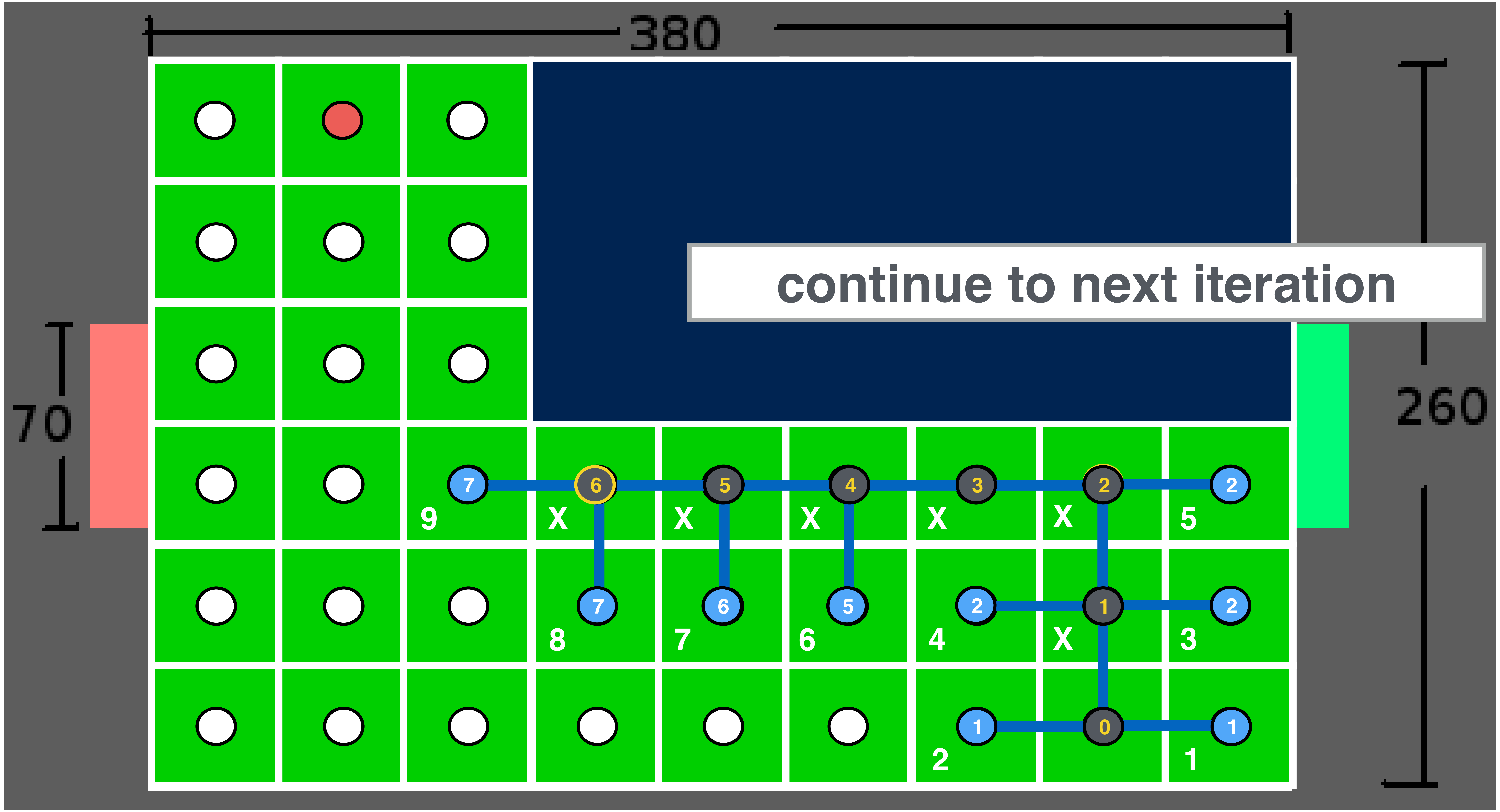


Depth-first search



Depth-first search



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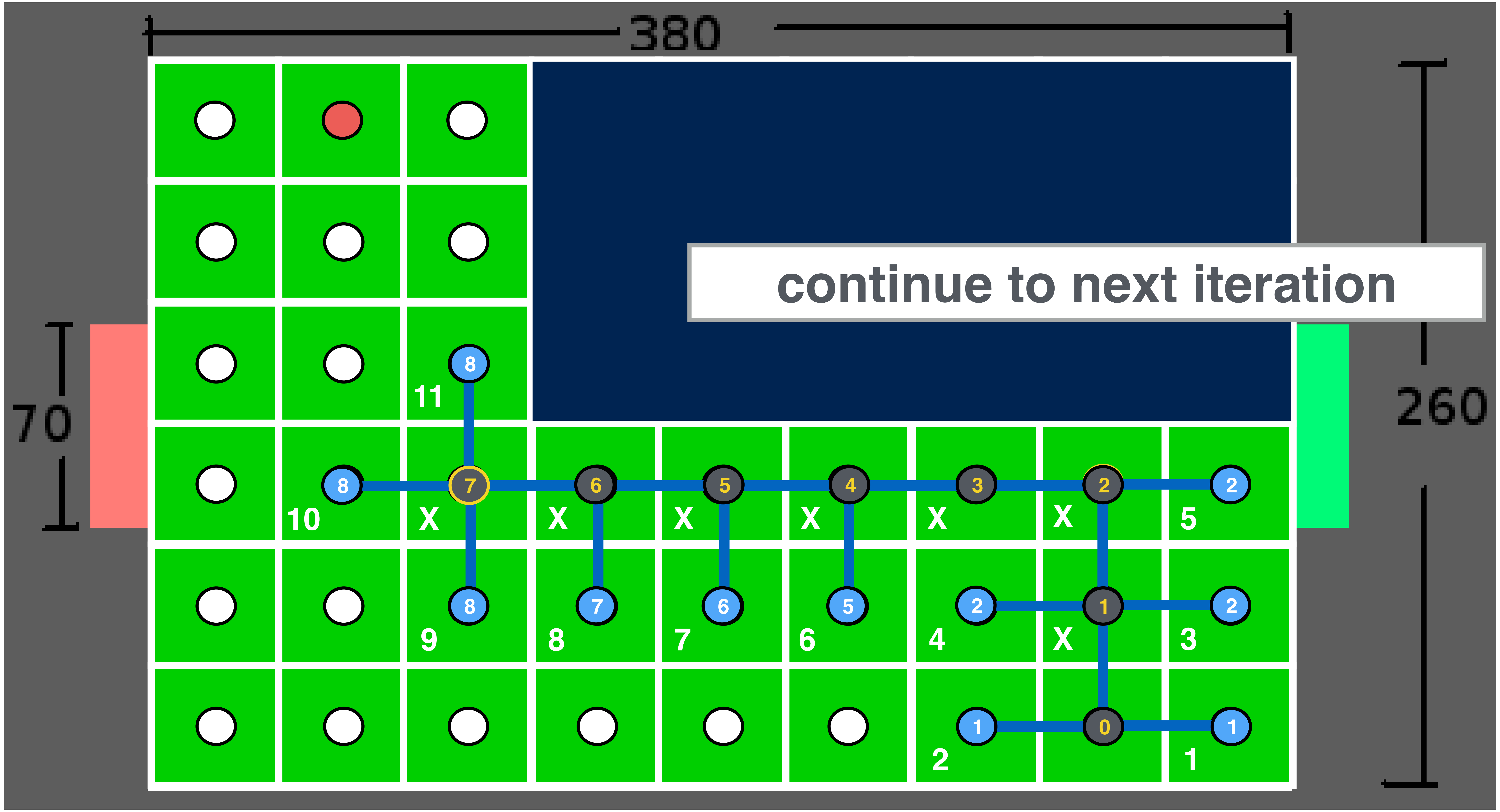

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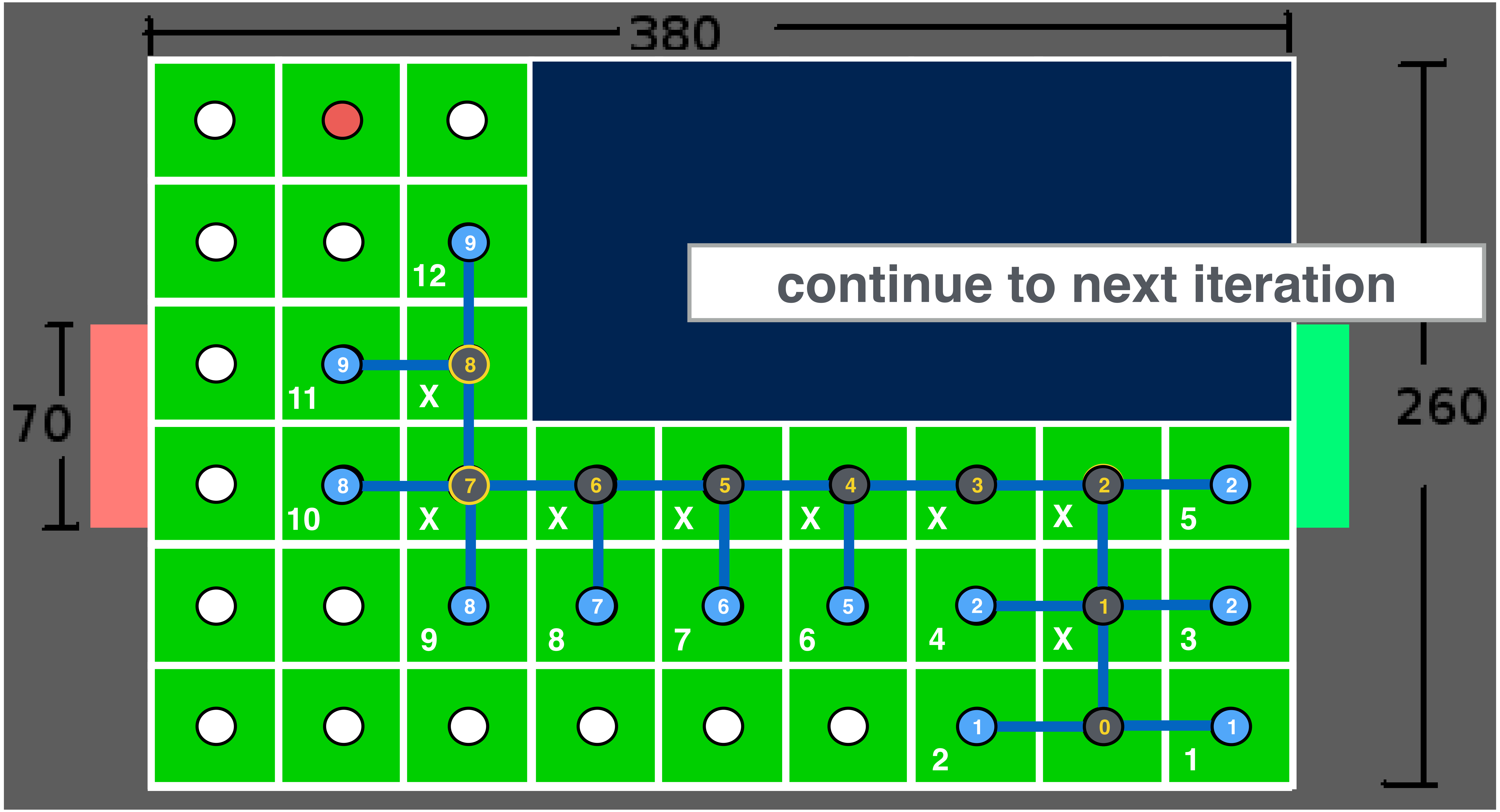
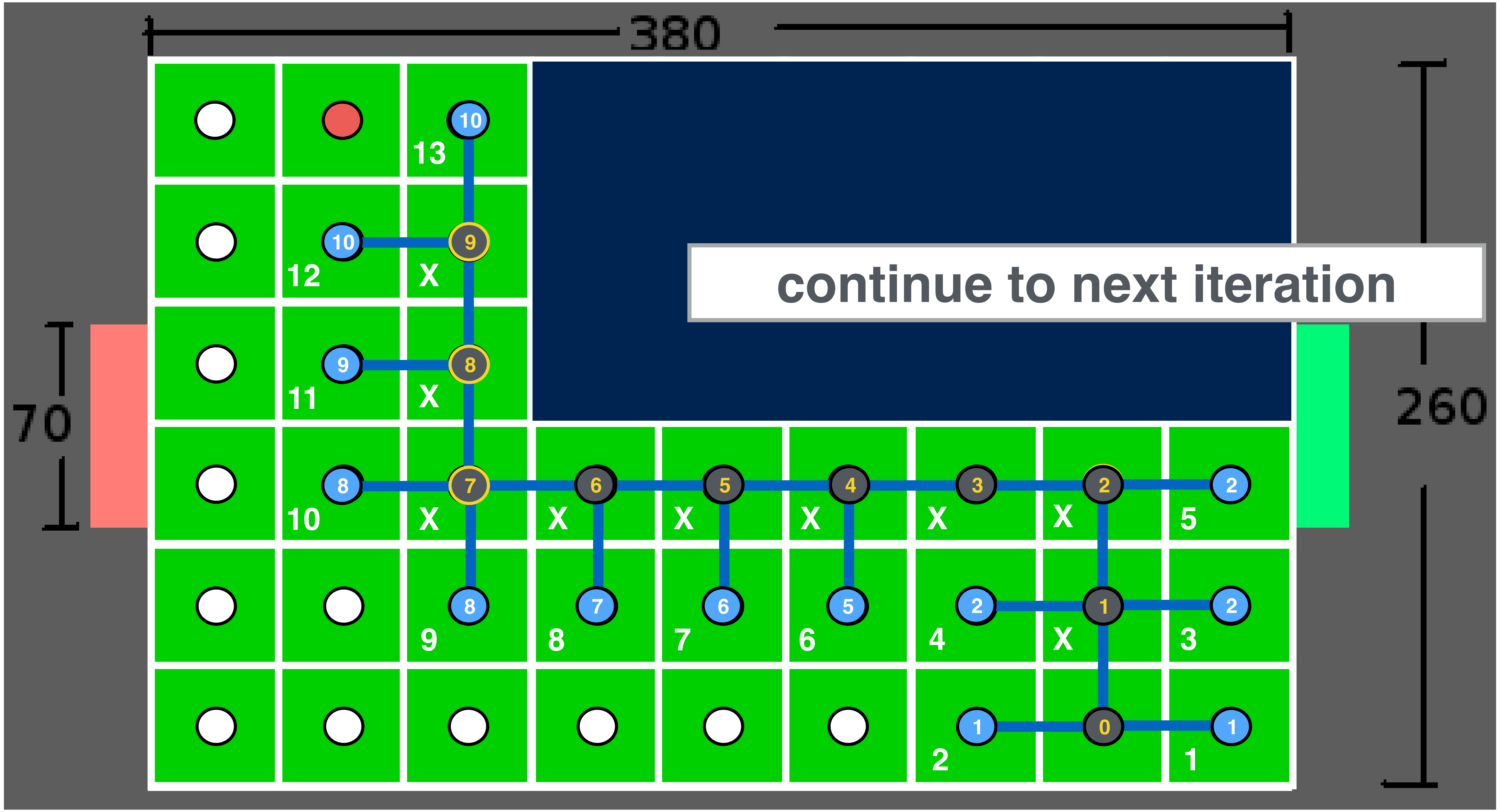


Depth-first search



Depth-first search

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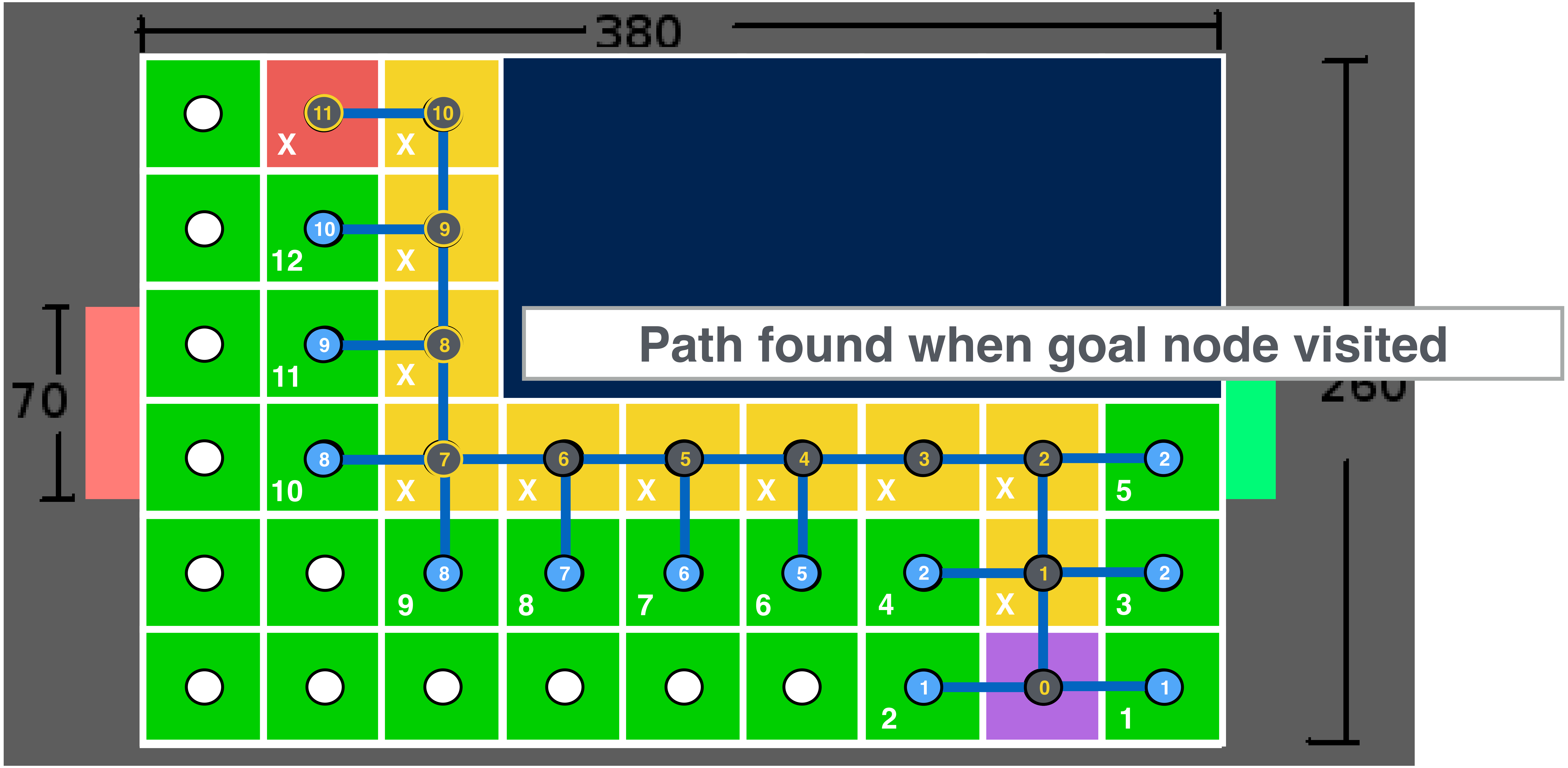


Let’s turn this idea into code

<u>Search algorithm template</u>

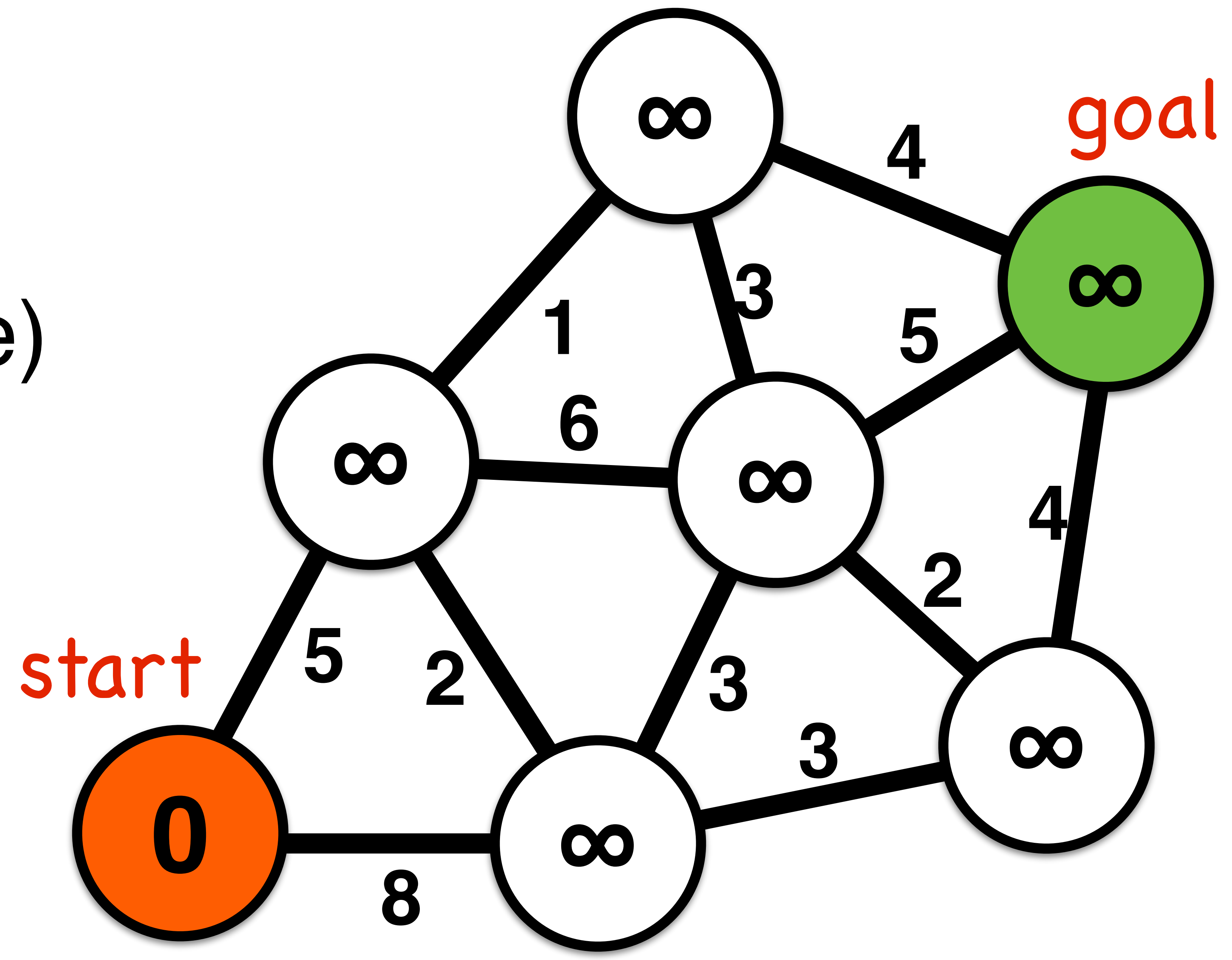
all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

```
    while visit_list != empty && current_node != goal
      cur_node ← highestPriority(visit_list)
      visited_cur_node ← true
      for each nbr in not_visited(adjacent(cur_node))
        add(nbr to visit_list)
        if dist_nbr > dist_cur_node + distStraightLine(nbr,cur_node)
          parent_nbr ← current_node
          dist_nbr ← dist_cur_node + distStraightLine(nbr,cur_node)
        end if
      end for loop
    end while loop
```

output ← parent, distance



Search algorithm template

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start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

while visit_list != empty && current_node != goal

Initialization

- each node has a distance and a parent
  - distance: distance along route from start
  - parent: routing from node to start
- visit a chosen start node first
- all other nodes are unvisited and have high distance

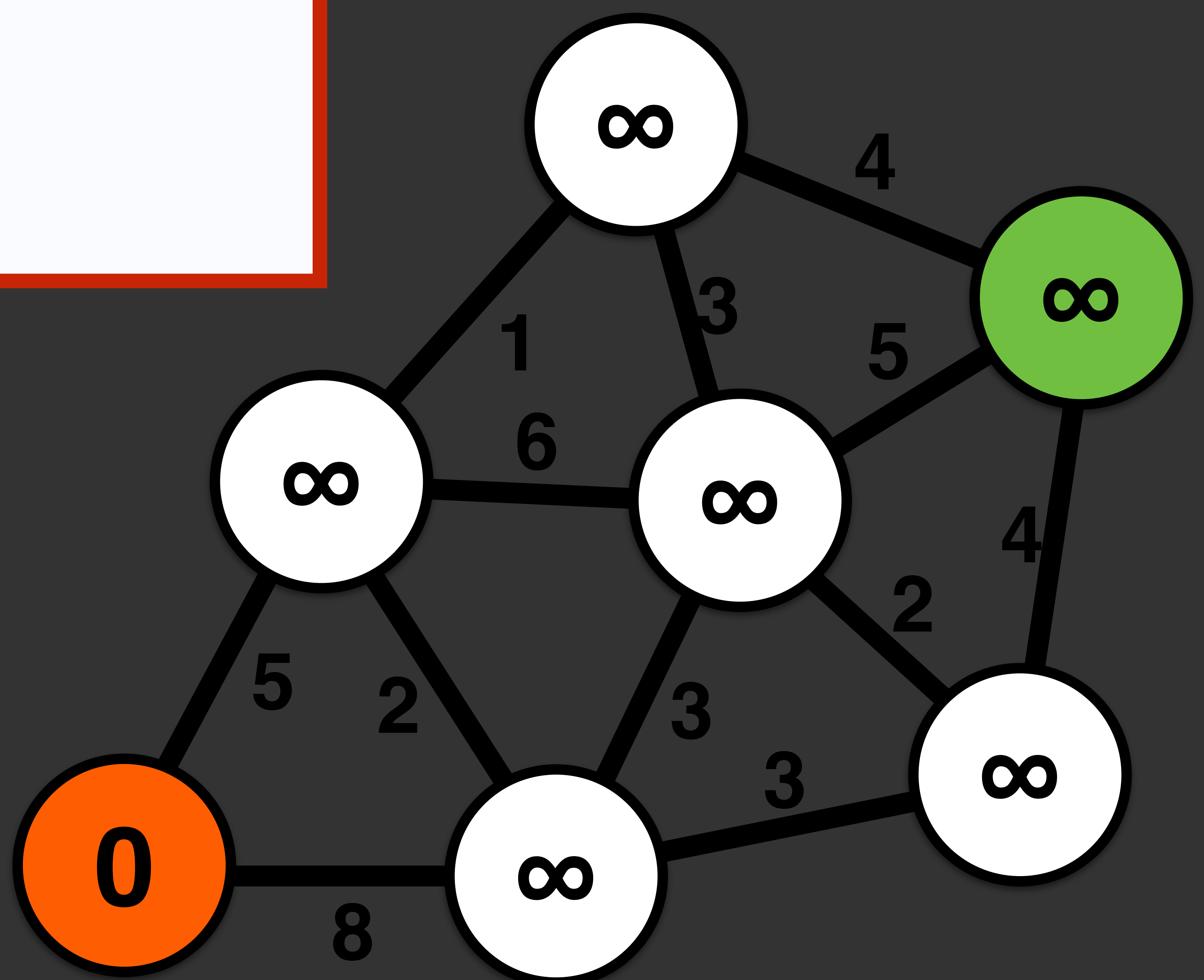
$dist_{nbr}$ ← $dist_{cur\_node}$ + distStraightLine(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance



Search algorithm template

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start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

while visit_list != empty && current_node != goal

cur_node ← highestPriority(visit_list)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

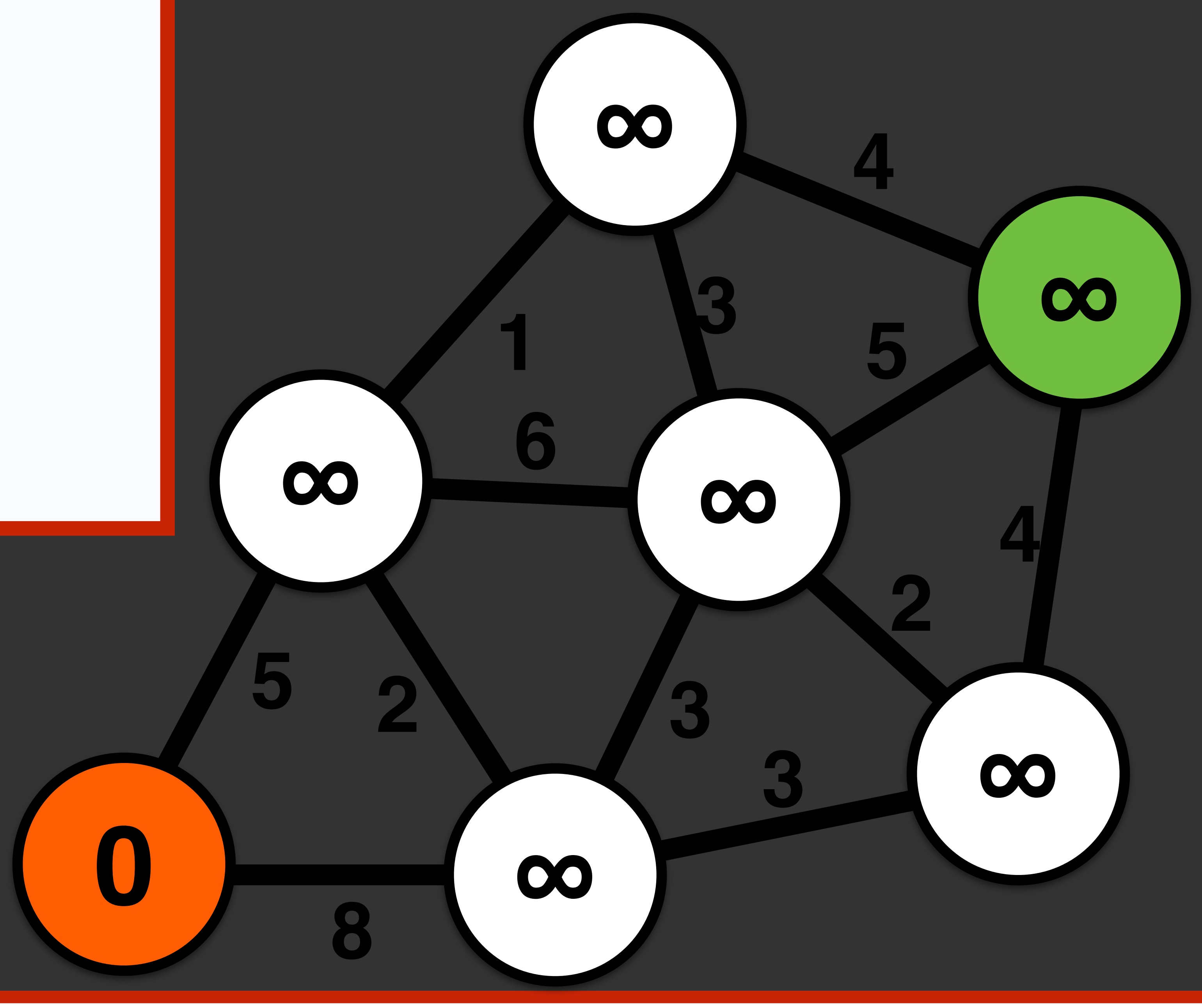
Main Loop

- visits every node to compute its distance and parent
- at each iteration:
  - select the node to visit based on its priority
  - remove current node from visit_list

end for loop

end while loop

output ← parent, distance



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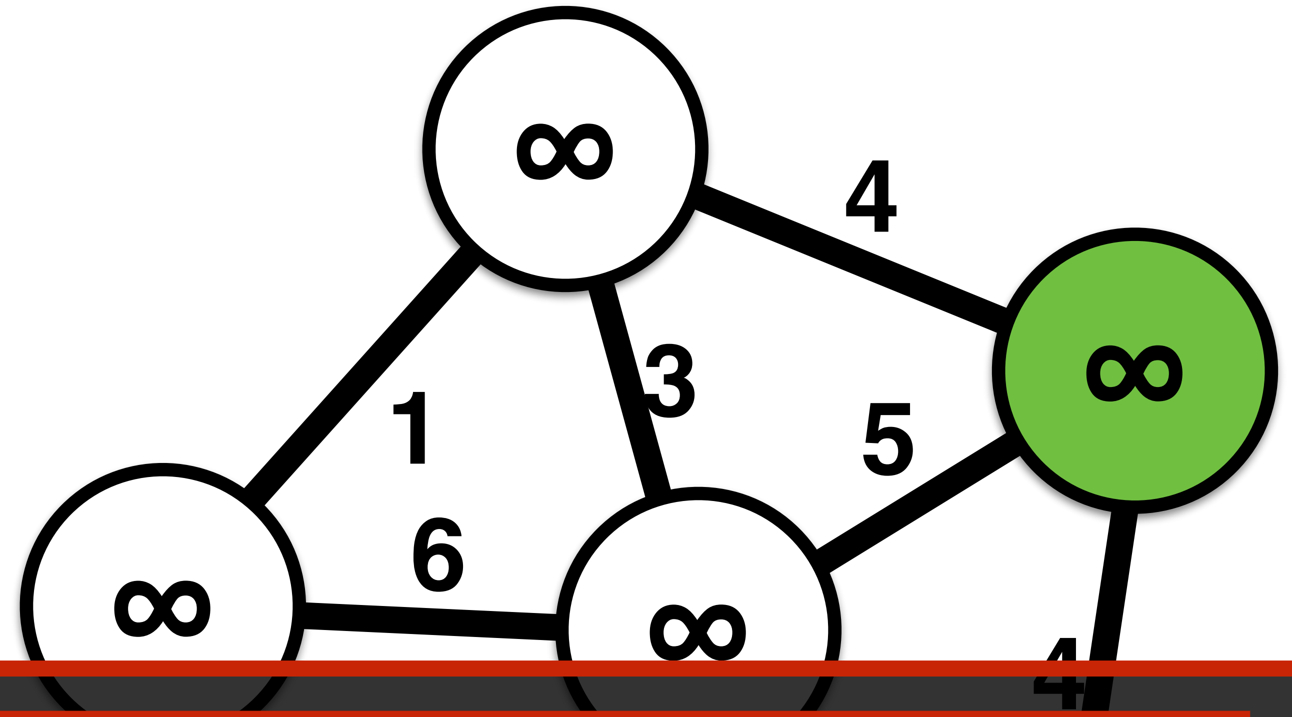
start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

```
    while visit_list != empty && current_node != goal
        cur_node ← highestPriority(visit_list)
        visited_cur_node ← true
        for each nbr in not_visited(adjacent(cur_node))
            add(nbr to visit_list)
            if dist_nbr > dist_cur_node + distStraightLine(nbr,cur_node)
                parent_nbr ← current_node
                dist_nbr ← dist_cur_node + distStraightLine(nbr,cur_node)
            end if
```



For each iteration on a single node

- add all unvisited neighbors of the node to the visit list
- assign node as a parent to a neighbor, if it creates a shorter route

Search algorithm template

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

while visit_list != empty && current_node != goal

cur_node ← highestPriority(visit_list)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

add(nbr to visit_list)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

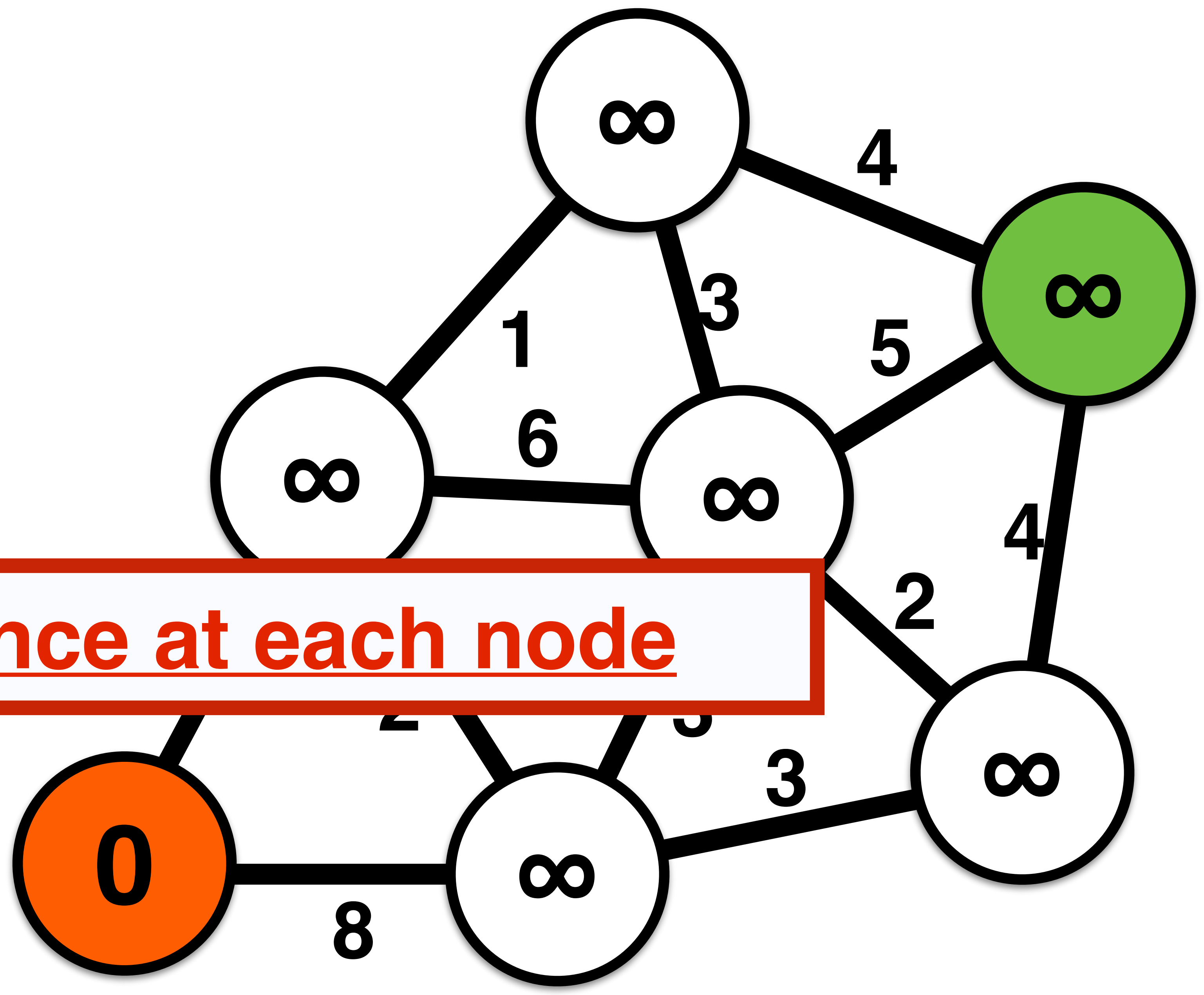
end if

end for loop

end while loop

output ← parent, distance

Output the resulting routing and path distance at each node



Depth-first search

Search algorithm template

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

while visit_list != empty && current_node != goal

cur_node ← **highestPriority**(visit_list)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

add(nbr to visit_list)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

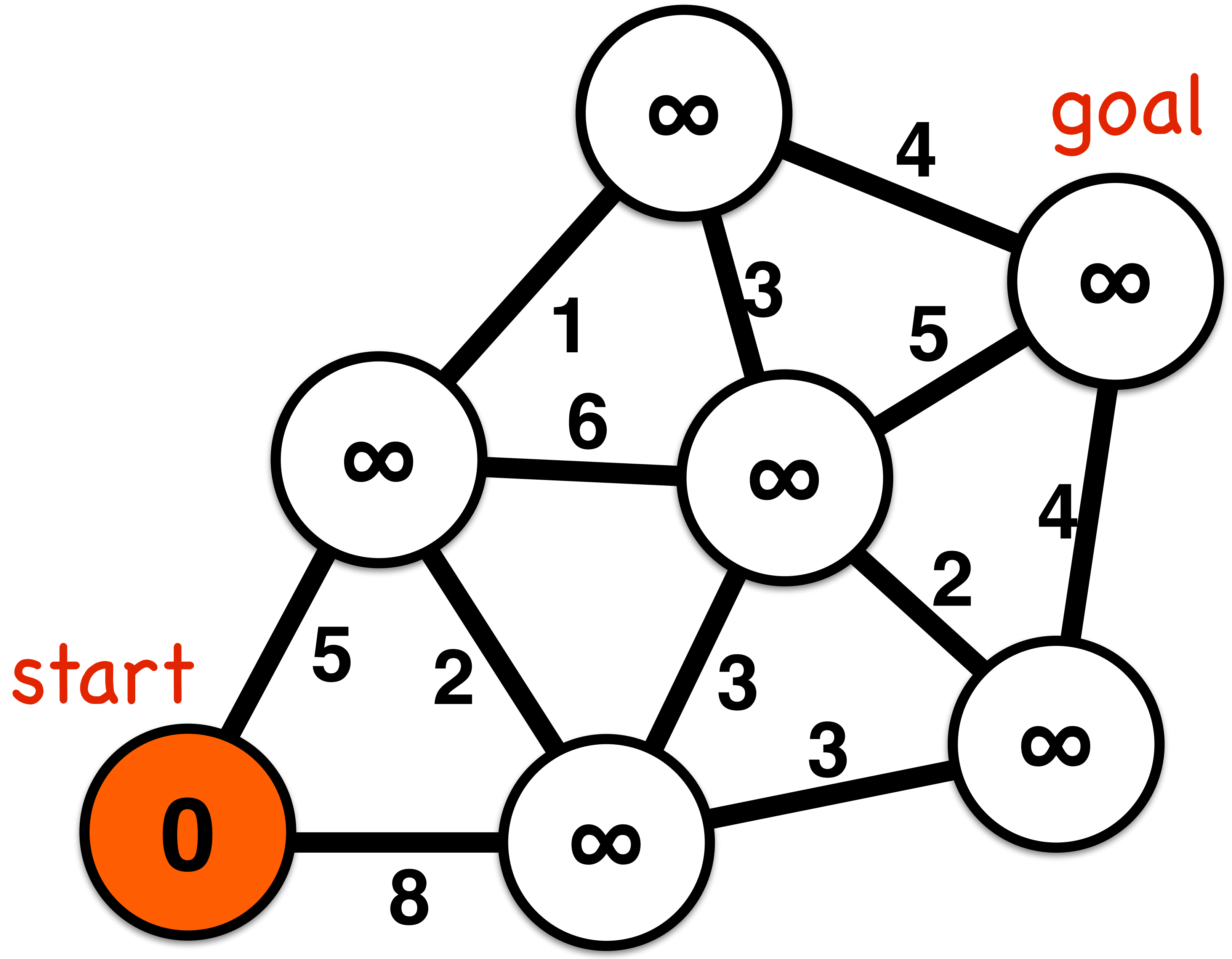
$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance



Depth-first search

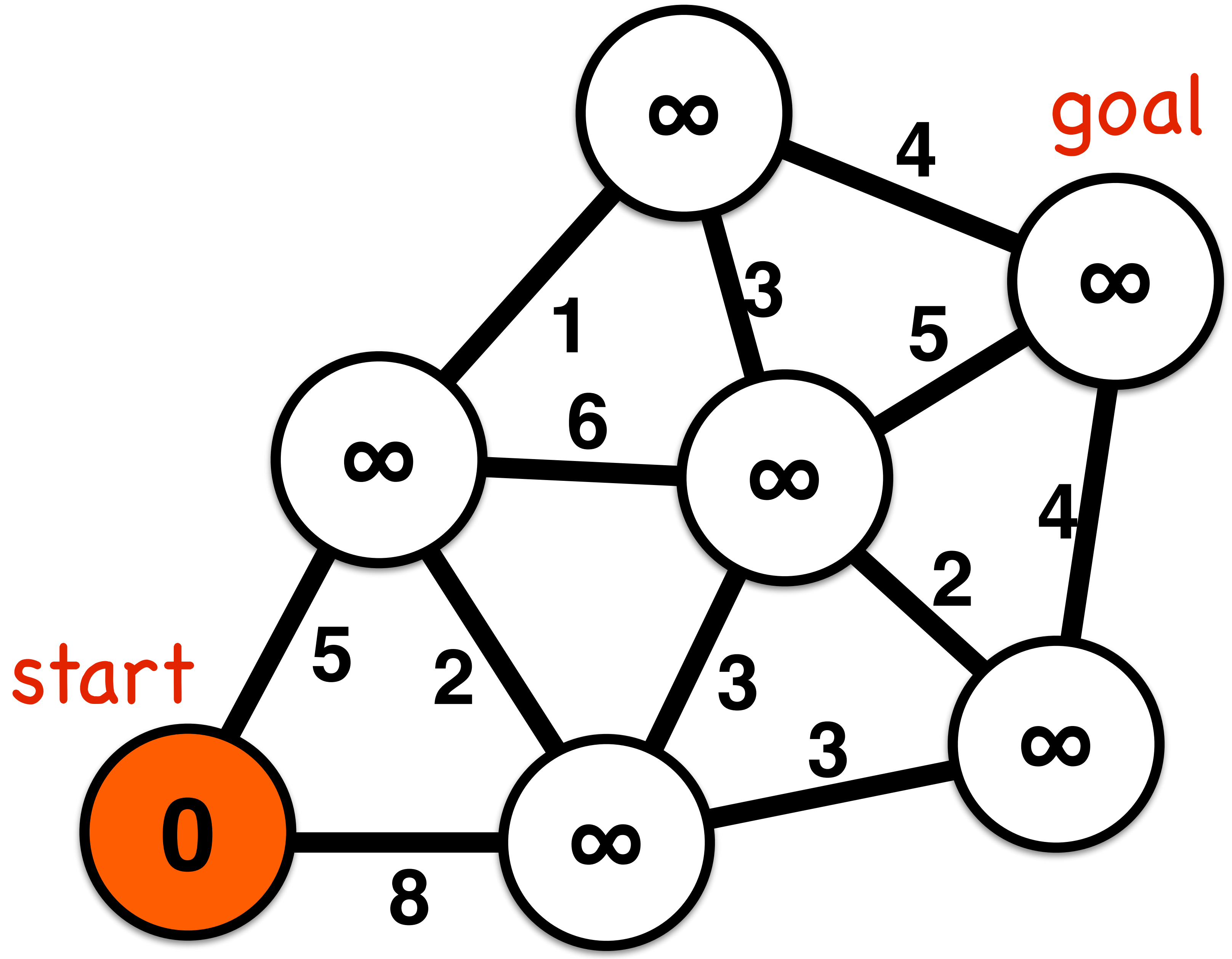
all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_stack ← start_node

```
    while visit_stack != empty && current_node != goal
        cur_node ← pop(visit_stack)
        visited_cur_node ← true
        for each nbr in not_visited(adjacent(cur_node))
            push(nbr to visit_stack)
            if dist_nbr > dist_cur_node + distance(nbr,cur_node)
                parent_nbr ← current_node
                dist_nbr ← dist_cur_node + distance(nbr,cur_node)
            end if
        end for loop
    end while loop
output ← parent, distance
```

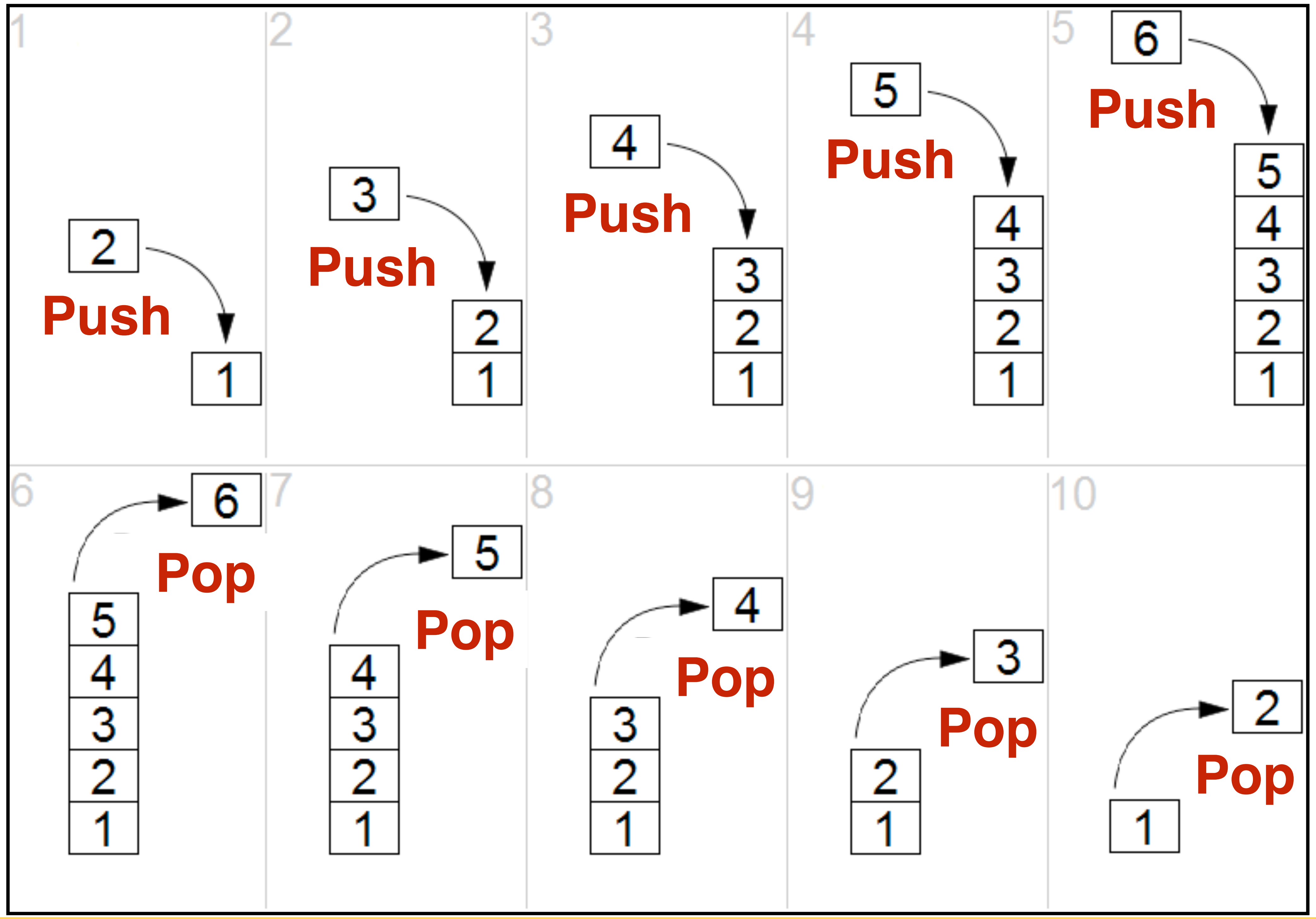
Priority:
Most recent

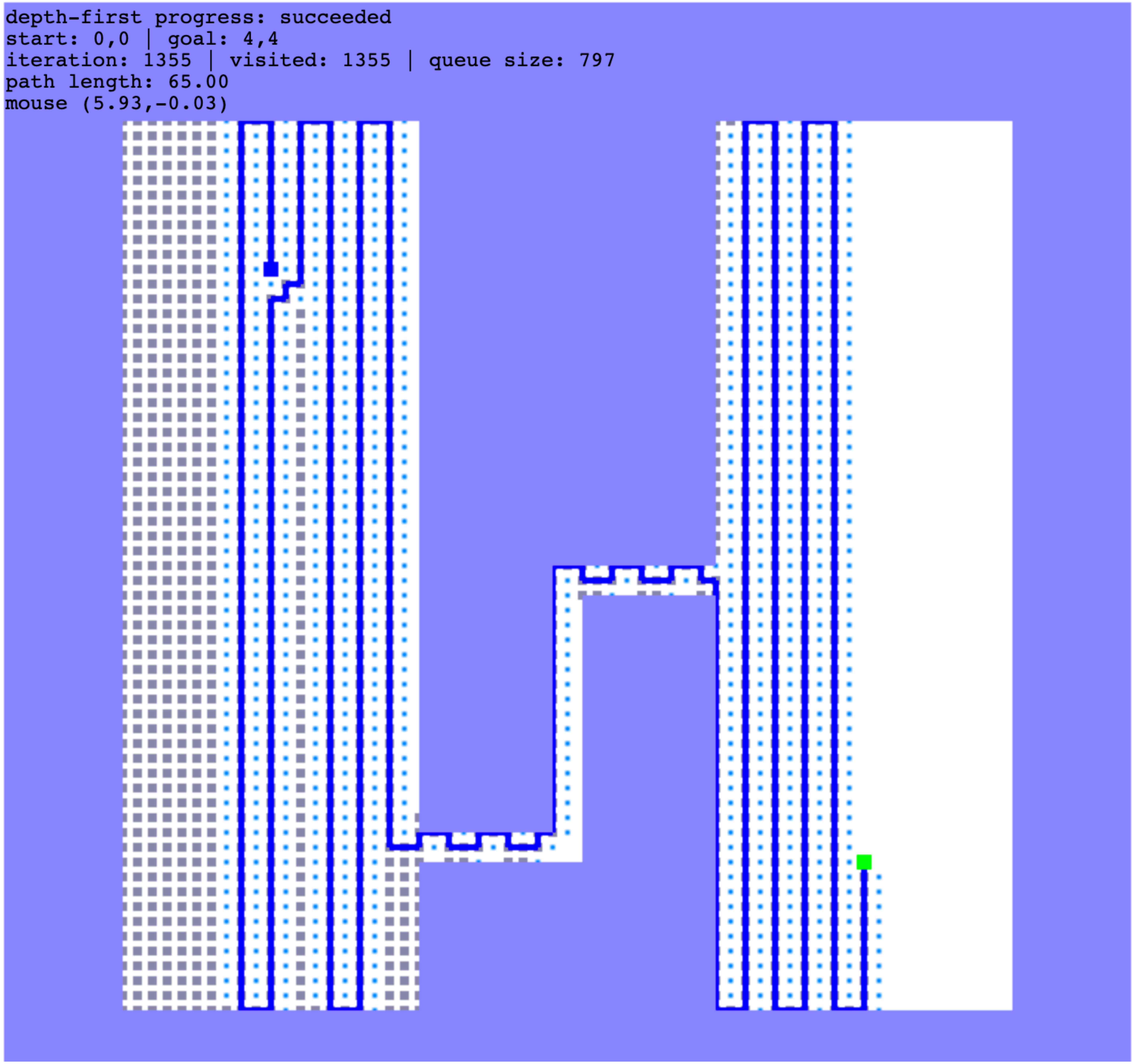


Stack data structure

A stack is a "last in, first out" (or LIFO) structure, with two operations:
push: to add an element to the top of the stack
pop: to remove and element from the top of the stack

Stack example for reversing the order of six elements




depth-first progress: succeeded
start: 0,0 | goal: 4,4
iteration: 1355 | visited: 1355 | queue size: 797
path length: 65.00
mouse (5.93,-0.03)

Breadth-first search

Search algorithm template

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

while visit_list != empty && current_node != goal

cur_node ← highestPriority(visit_list)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

add(nbr to visit_list)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

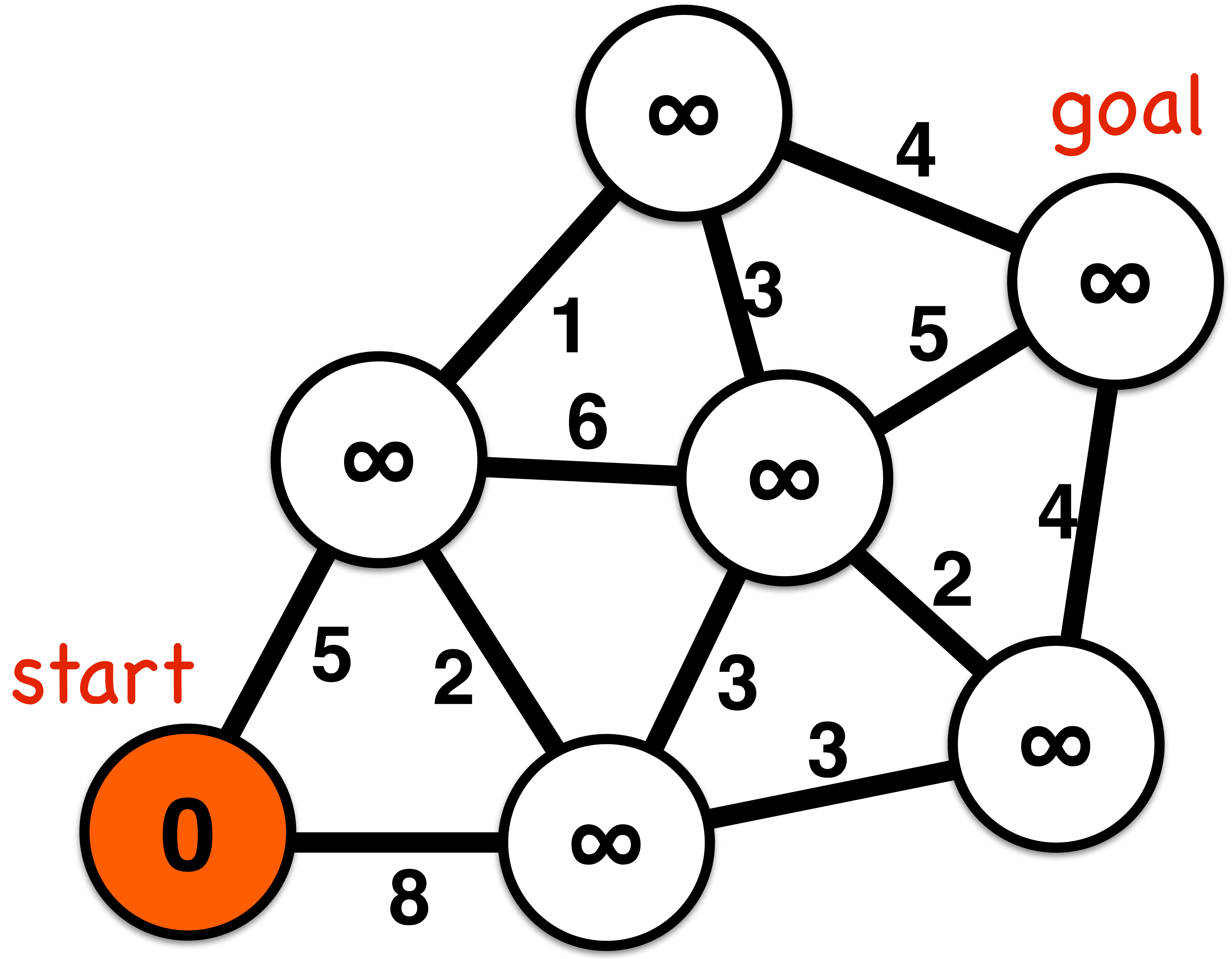
$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance



Breadth-first search

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

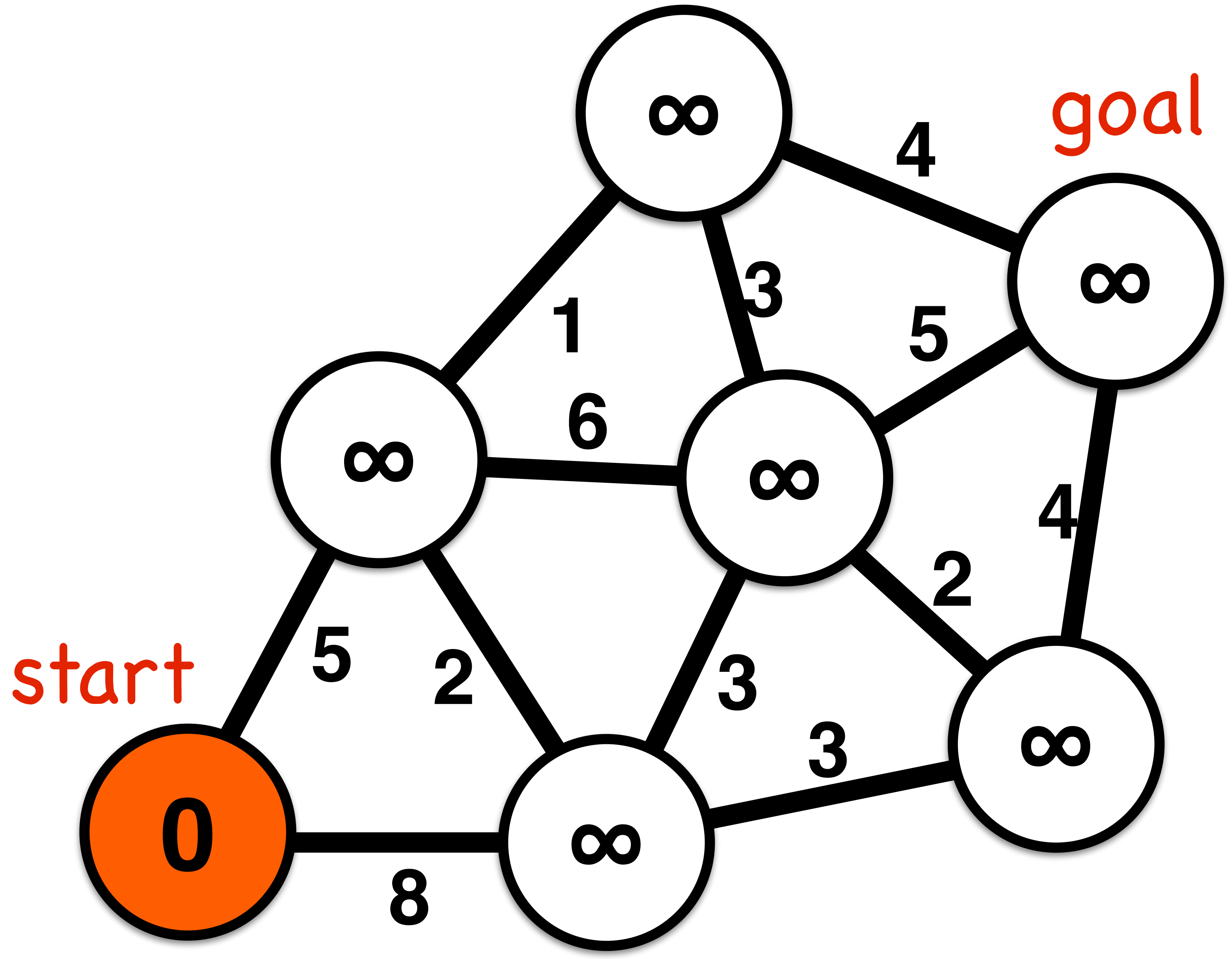
start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_queue ← start_node

```
while visit_queue != empty && current_node != goal
    cur_node ← dequeue(visit_queue)
    visited_cur_node ← true
    for each nbr in not_visited(adjacent(cur_node))
        enqueue(nbr to visit_queue)
        if dist_nbr > dist_cur_node + distance(nbr,cur_node)
            parent_nbr ← current_node
            dist_nbr ← dist_cur_node + distance(nbr,cur_node)
        end if
    end for loop
end while loop
```

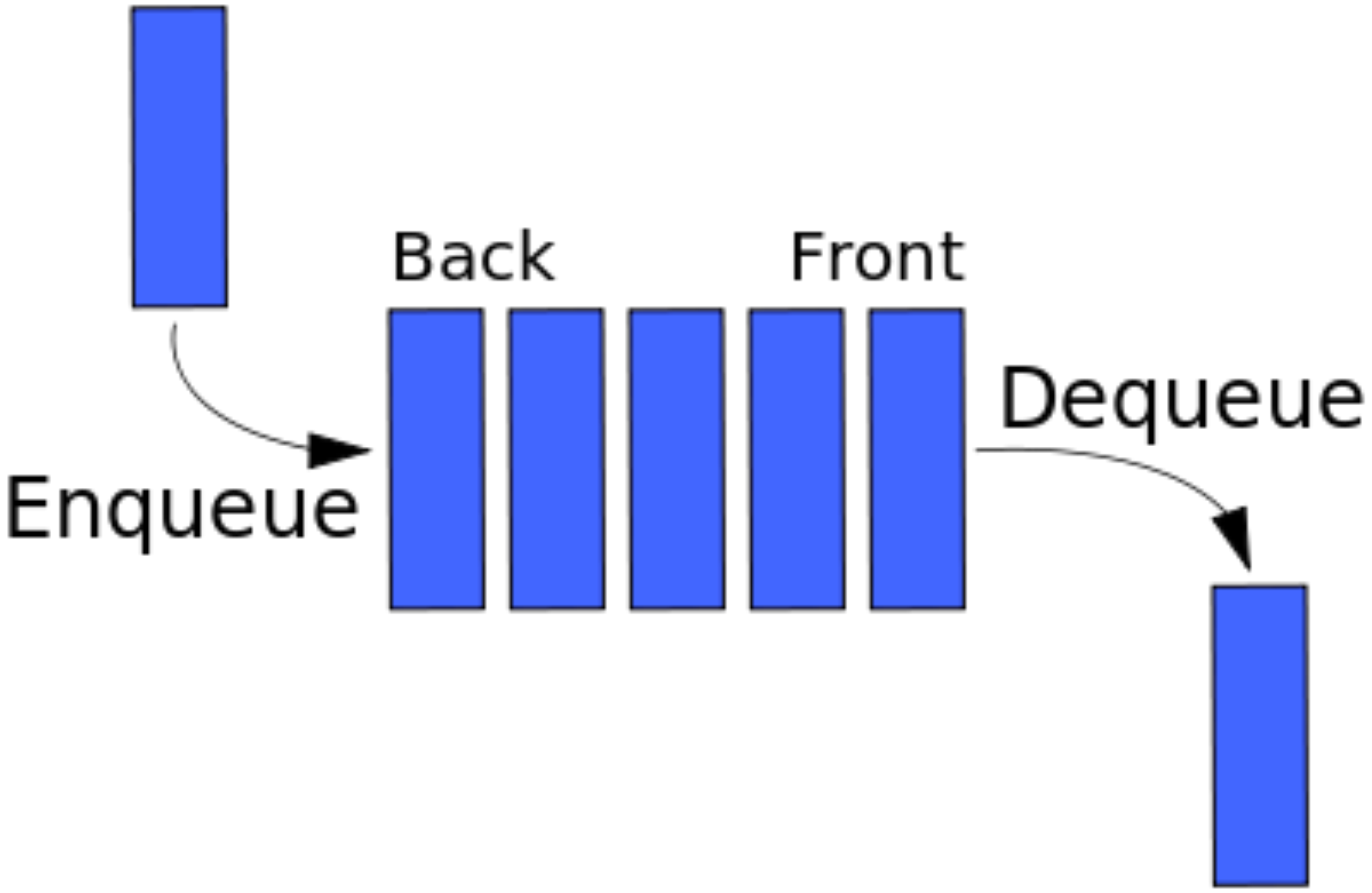
output ← parent, distance

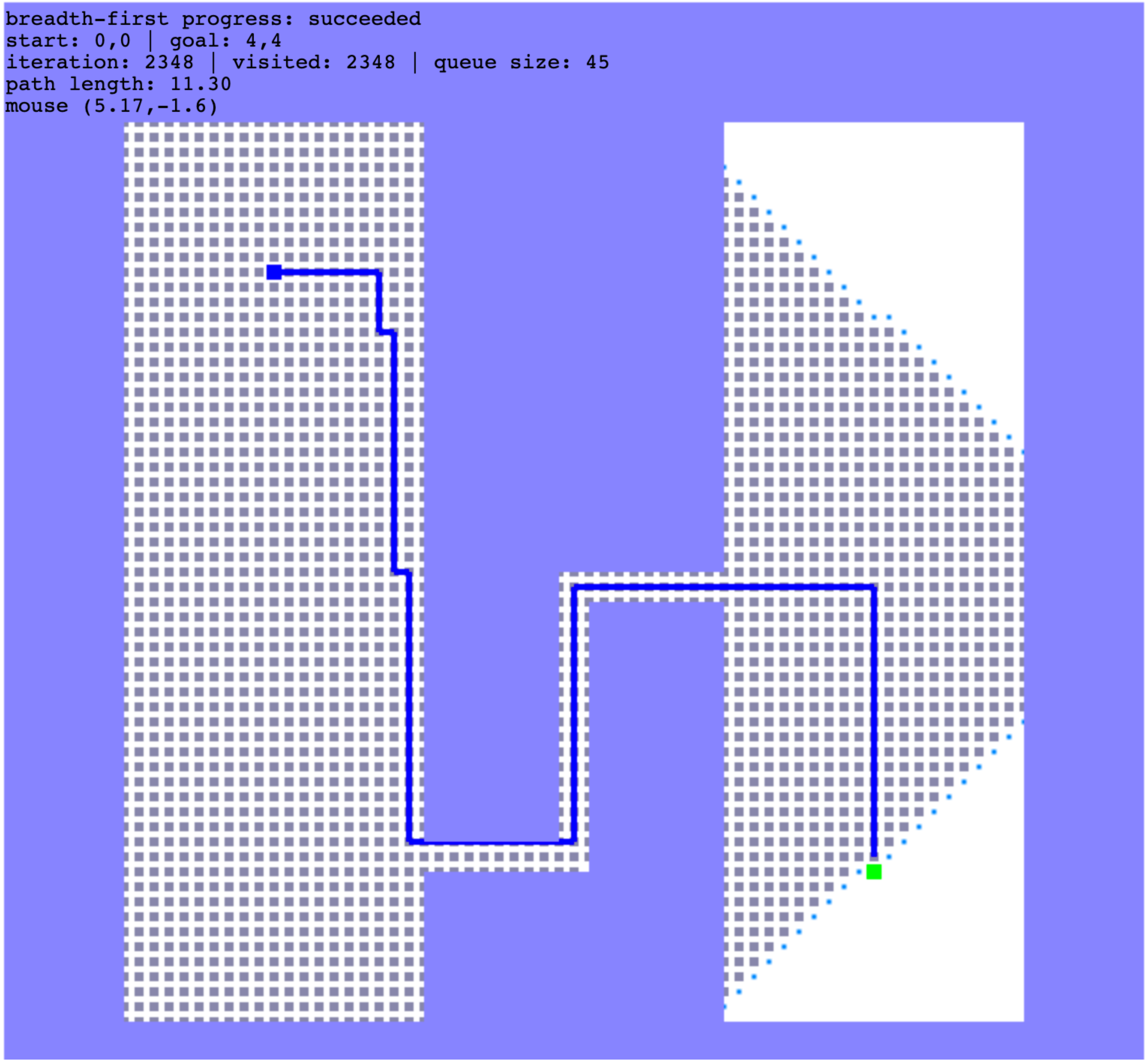
Priority:
Least recent



Queue data structure

A queue is a "first in, first out" (or FIFO) structure, with two operations
enqueue: to add an element to the back of the stack
dequeue: to remove an element from the front of the stack




breadth-first progress: succeeded
start: 0,0 | goal: 4,4
iteration: 2348 | visited: 2348 | queue size: 45
path length: 11.30
mouse (5.17,-1.6)

Dijkstra’s algorithm

Search algorithm template

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_list ← start_node

while visit_list != empty && current_node != goal

cur_node ← **highestPriority**(visit_list)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

add(nbr to visit_list)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

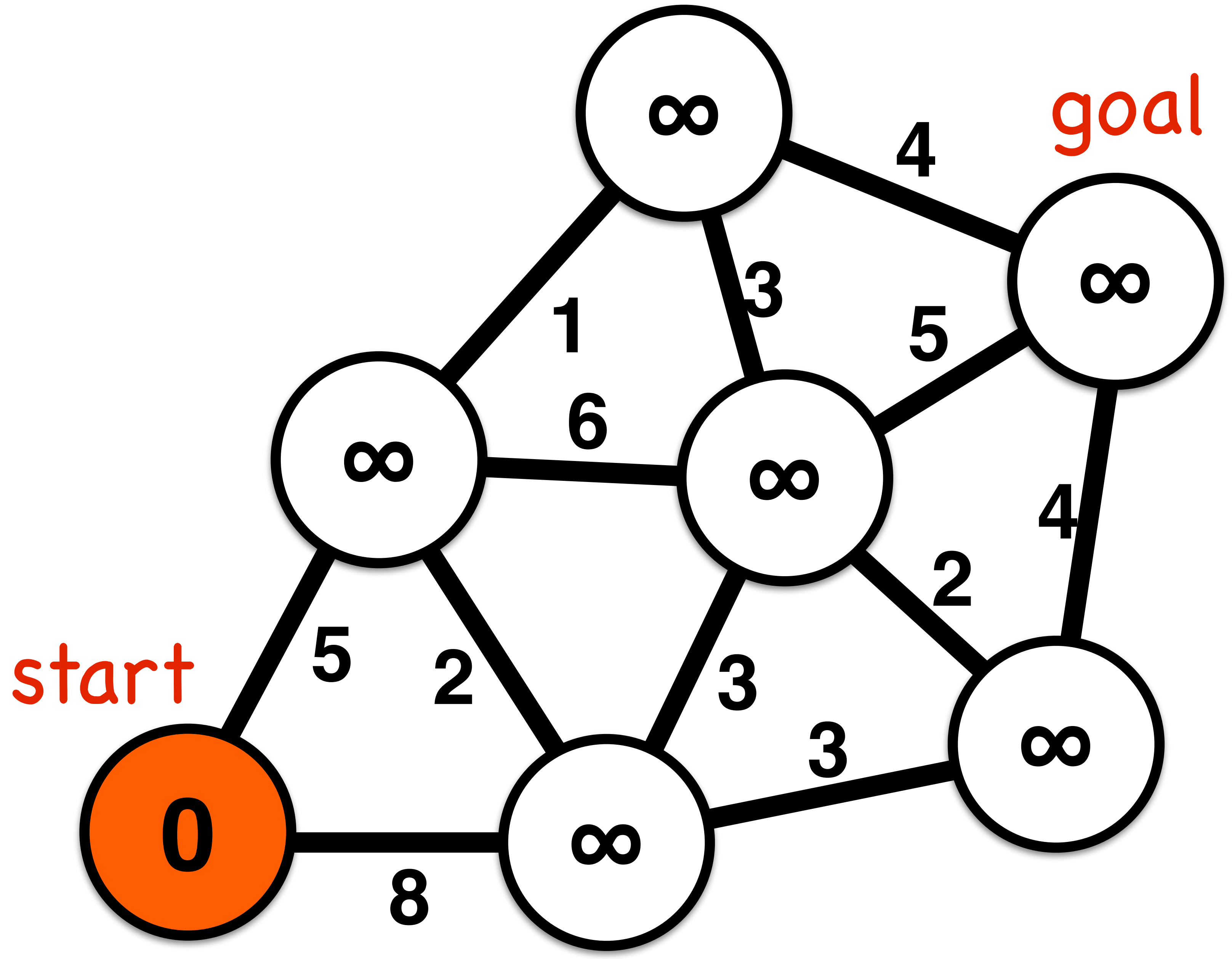
$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance



Dijkstra shortest path algorithm

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_queue ← start_node

while visit_queue != empty ~~&& current_node != goal~~

cur_node ← **min_distance**(**visit_queue**)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

enqueue(nbr to **visit_queue**)

if $dist_{nbr} > dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

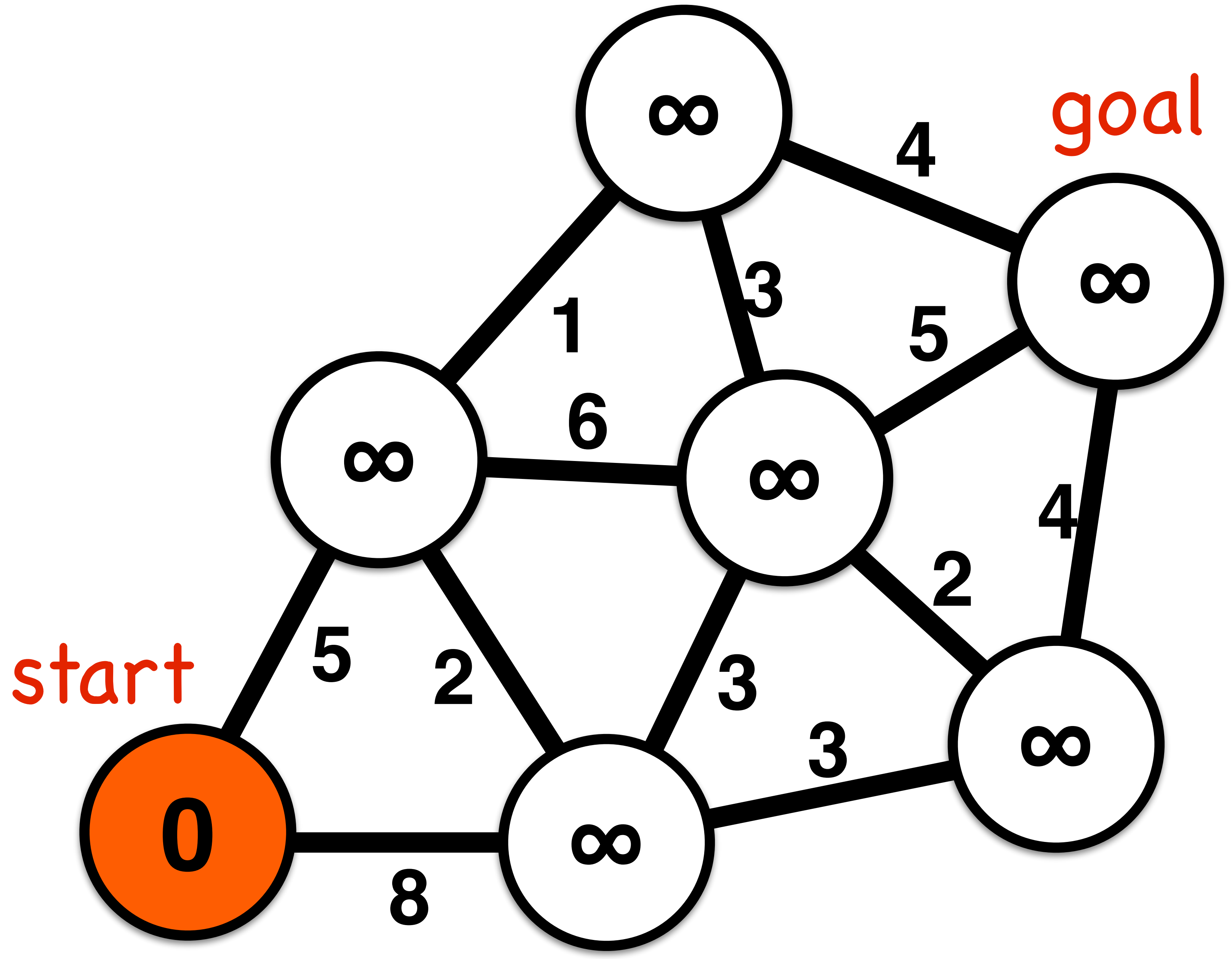
end if

end for loop

end while loop

output ← parent, distance

Priority:
Minimum route distance from start



Dijkstra shortest path algorithm

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visit_queue ← start_node

```
while visit_queue != empty ~~&& current_node != goal~~
    cur_node ← min_distance(visit_queue)
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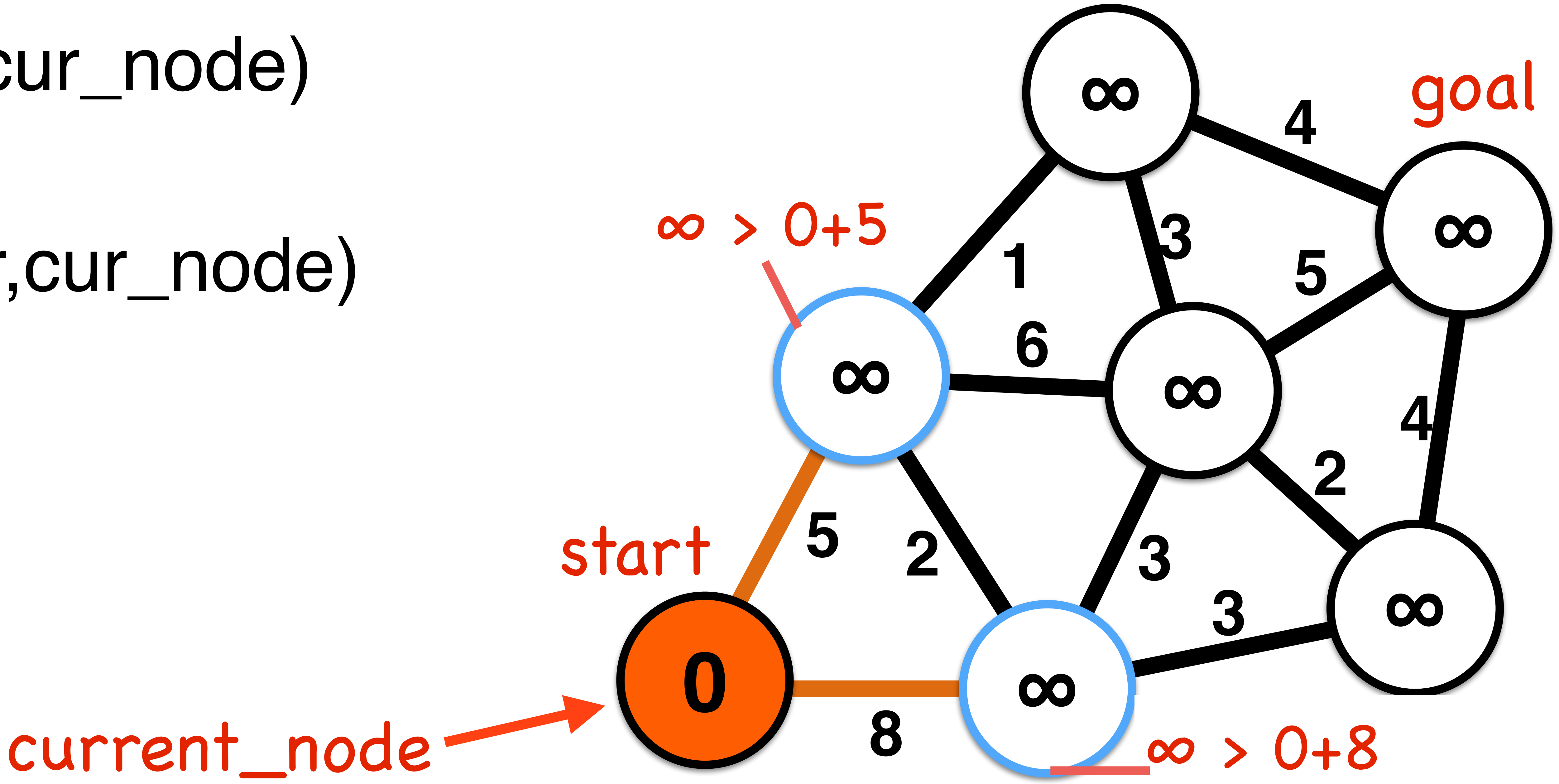
            parent_nbr ← current_node
            dist_nbr ← dist_cur_node + distance(nbr,cur_node)
        end if
    end for loop
end while loop
output ← parent, distance
```

Dijkstra walkthrough



Dijkstra shortest path algorithm

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visit_queue ← start_node

while visit_queue != empty ~~&& current_node != goal~~

cur_node ← min_distance(visit_queue)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

enqueue(nbr to visit_queue)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

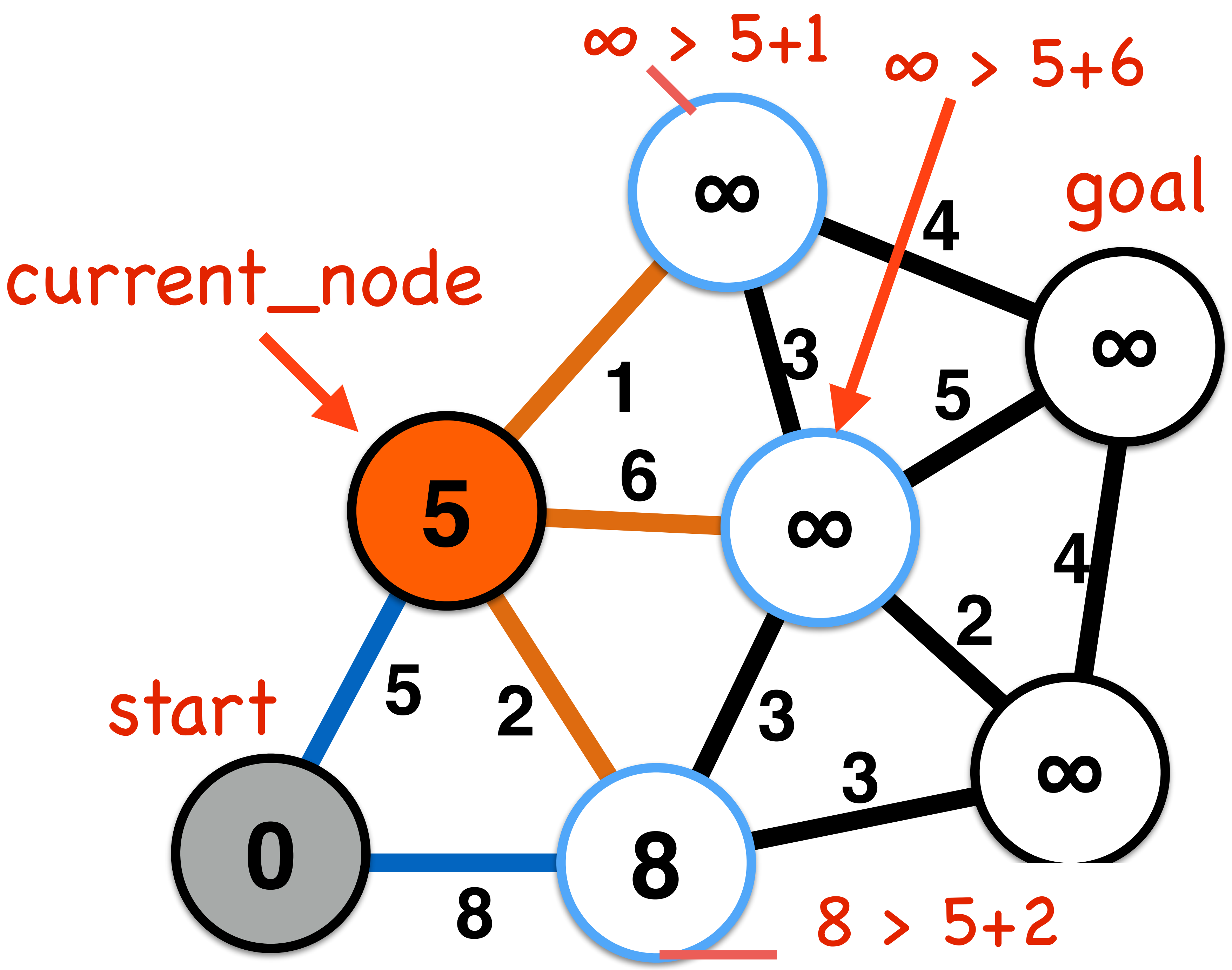
end if

end for loop

end while loop

output ← parent, distance

Dijkstra walkthrough



Dijkstra shortest path algorithm

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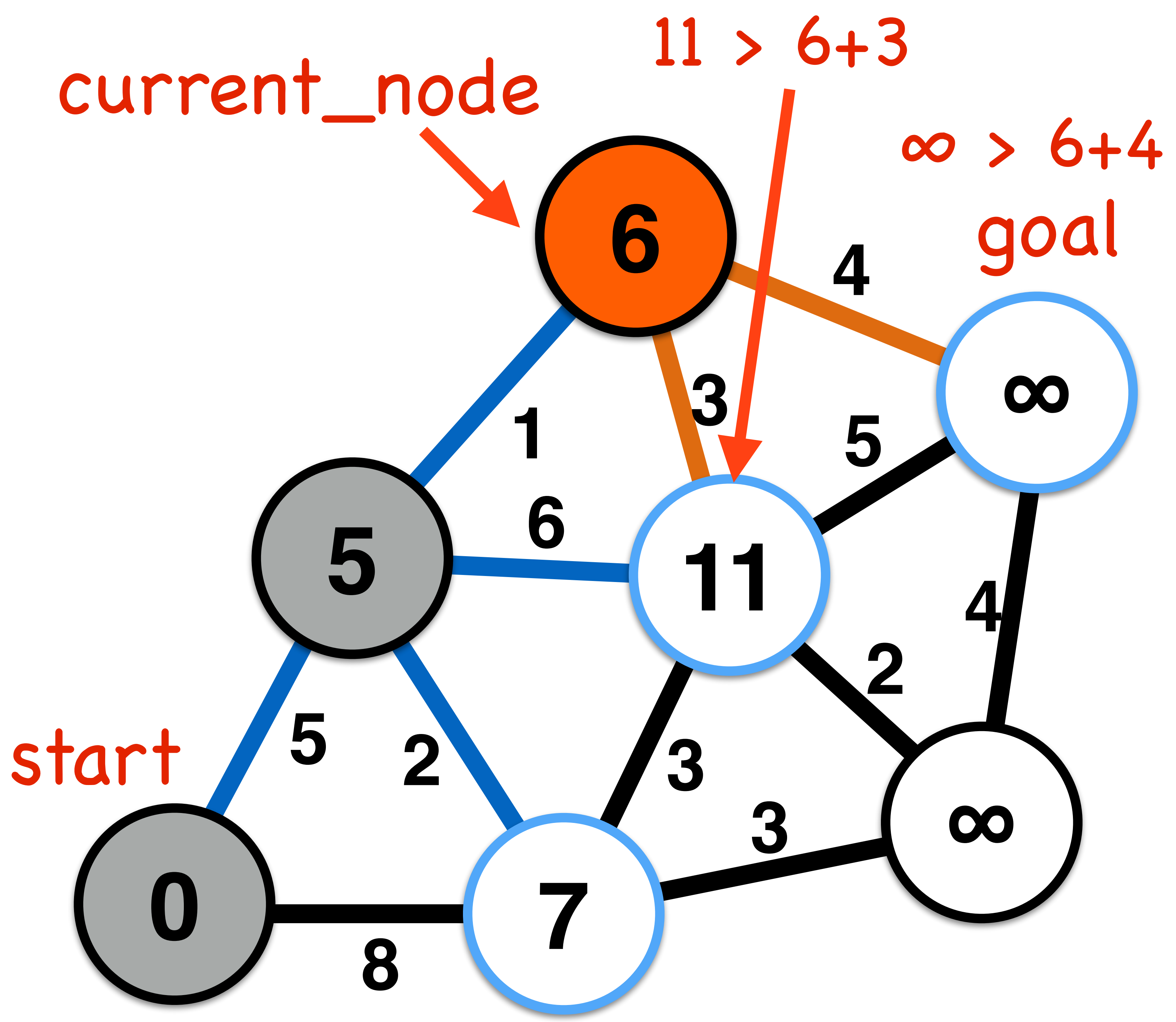
output ← parent, distance

Dijkstra walkthrough



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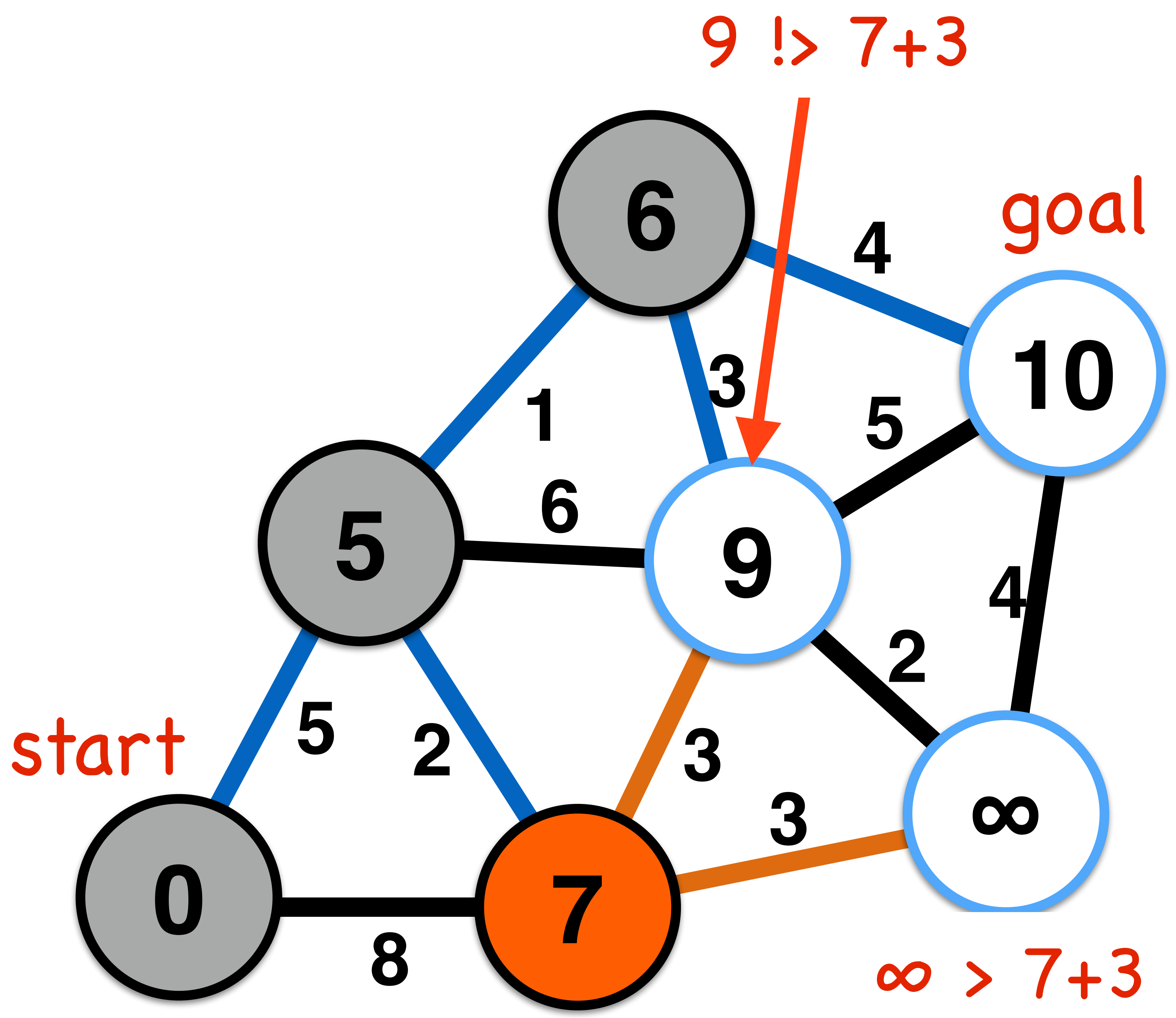
        end if
    end for loop
end while loop
```

output ← parent, distance



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visit_queue ← start_node

while visit_queue != empty ~~&& current_node != goal~~

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enqueue(nbr to visit_queue)

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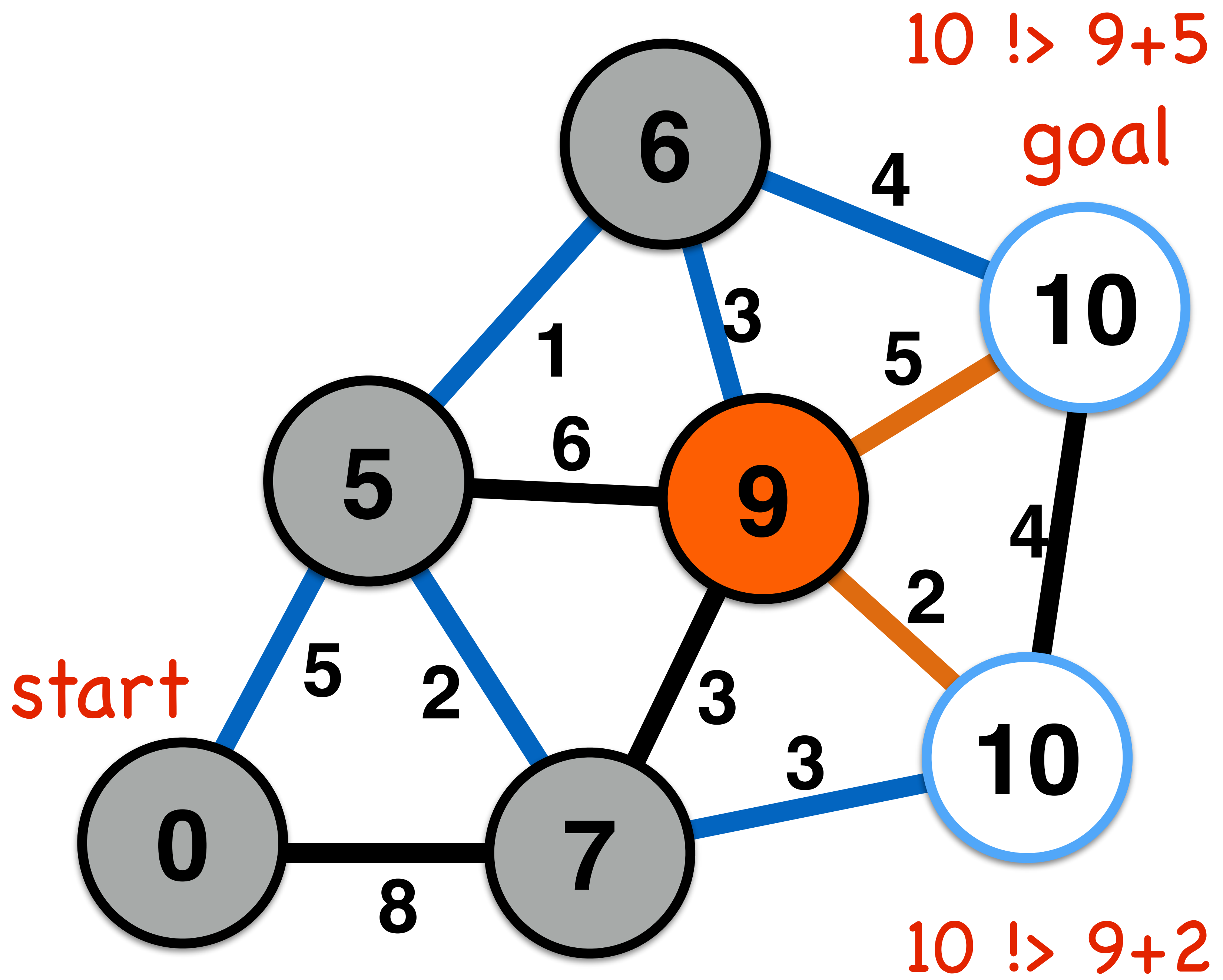
$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance



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for each nbr in not_visited(adjacent(cur_node))

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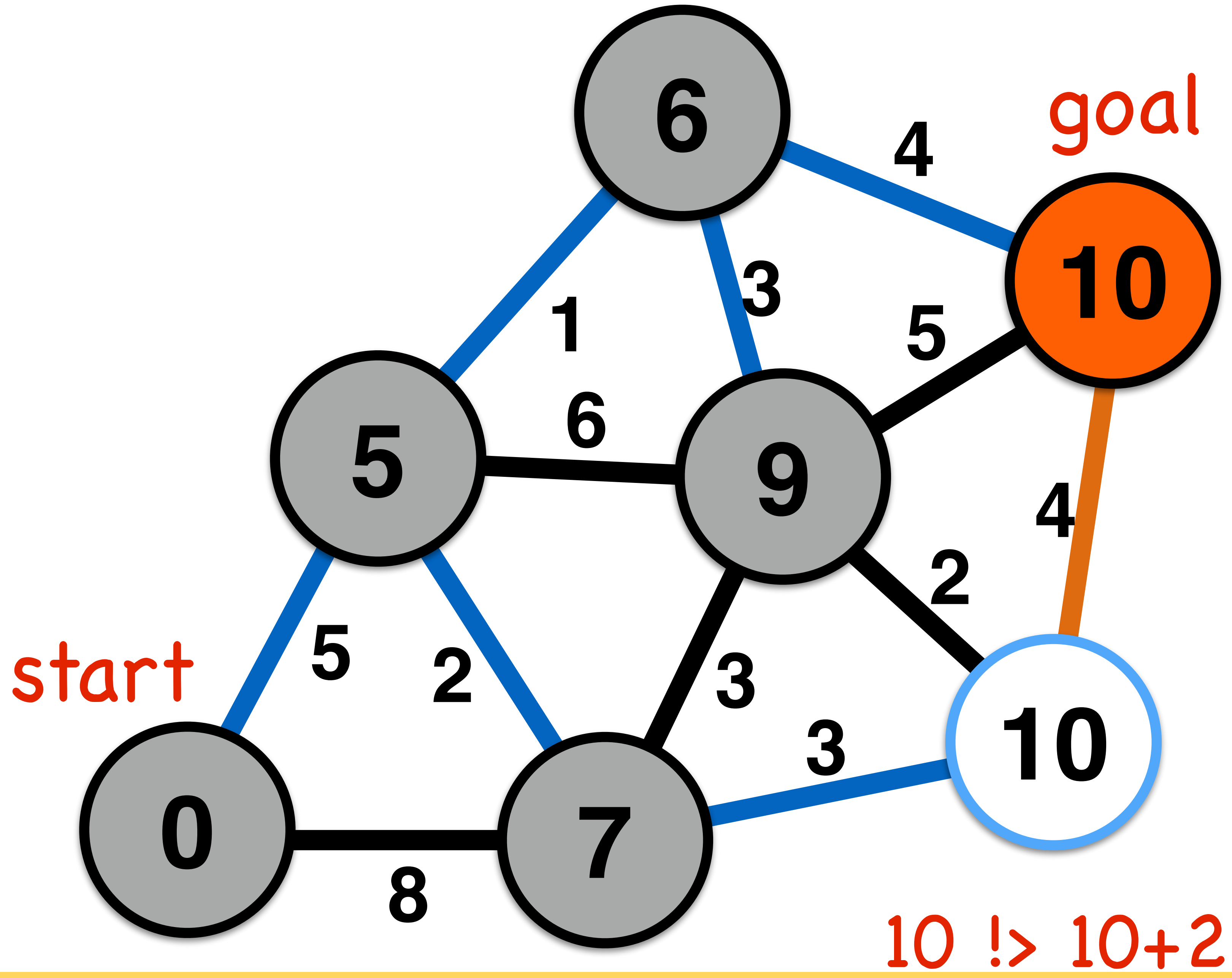
$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

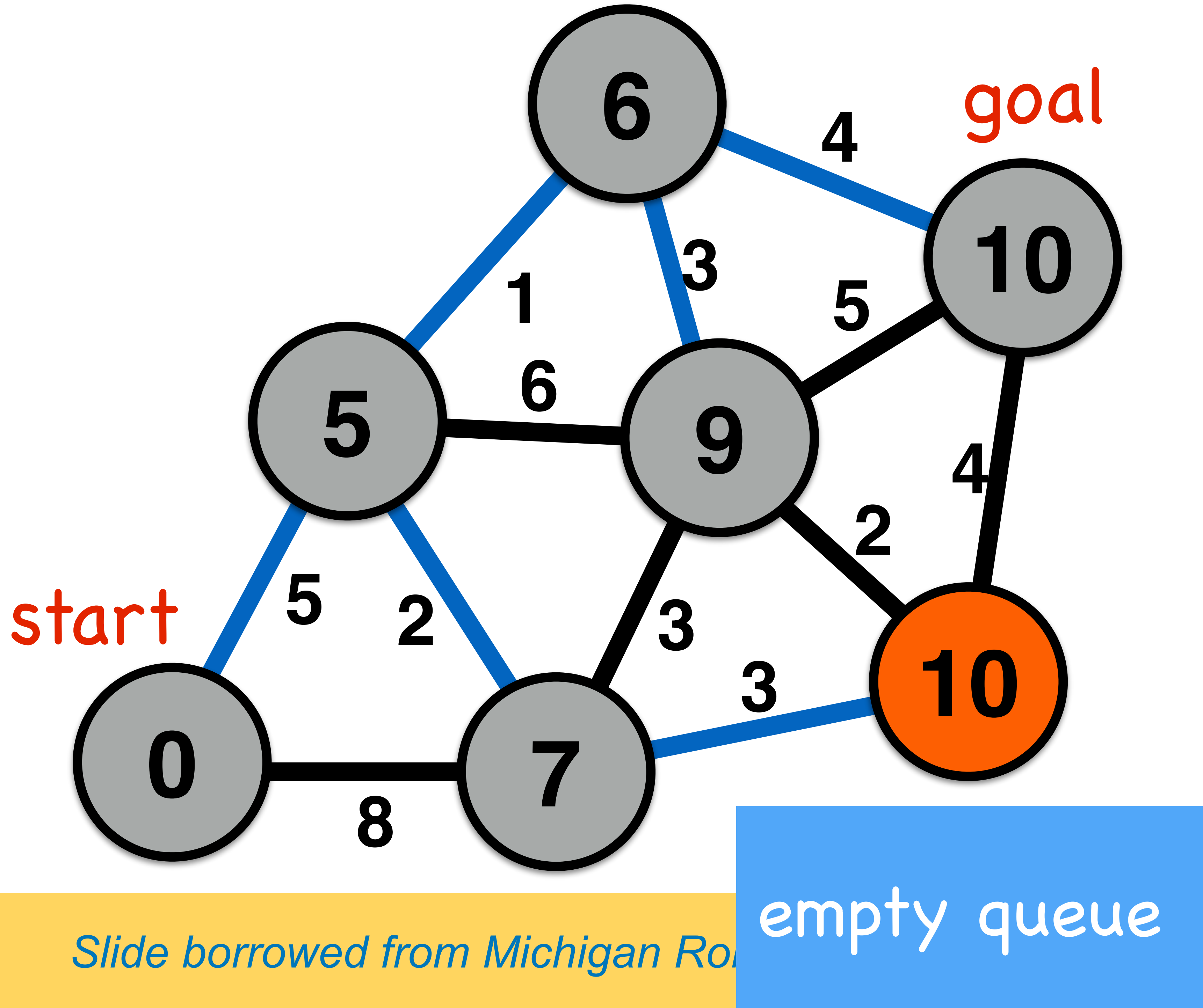
end while loop

output ← parent, distance



Dijkstra shortest path algorithm

```
all nodes ← {dist_start ← infinity, parent_start ← none, visited_start ← false}
start_node ← {dist_start ← 0, parent_start ← none, visited_start ← true}
visit_queue ← start_node
    while visit_queue != empty ~~&& current_node != goal~~
      cur_node ← min_distance(visit_queue)
      visited_cur_node ← true
      for each nbr in not_visited(adjacent(cur_node))
        enqueue(nbr to visit_queue)
        if dist_nbr > dist_cur_node + distance(nbr,cur_node)
          parent_nbr ← current_node
          dist_nbr ← dist_cur_node + distance(nbr,cur_node)
        end if
      end for loop
    end while loop
output ← parent, distance
```



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$parent_{nbr}$ ← current_node

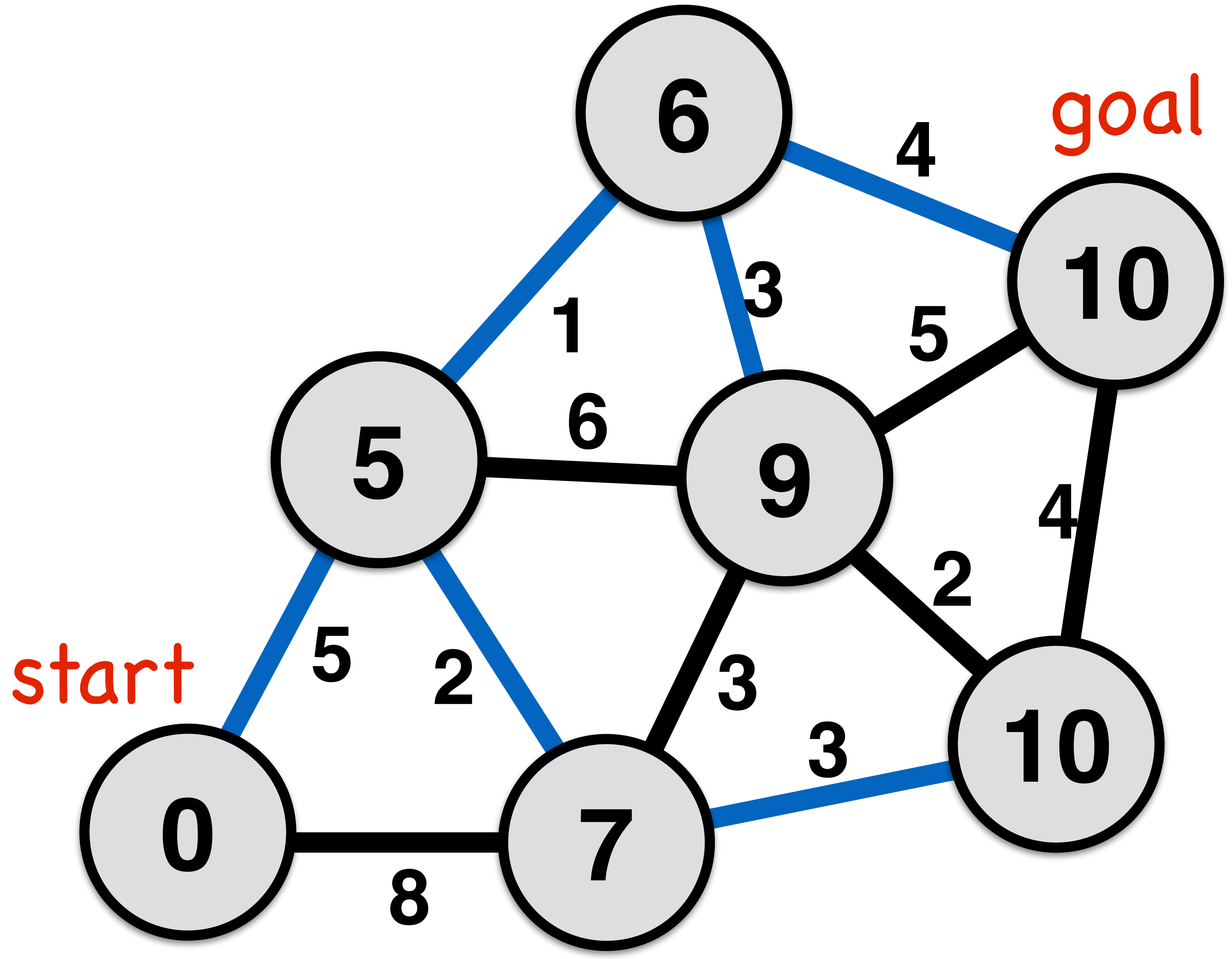
$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance



Dijkstra shortest path algorithm

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

What will search with Dijkstra's algorithm look like in this case?

visit_queue ← start_node

while visit_queue != empty ~~&& current_node != goal~~

cur_node ← min_distance(visit_queue)

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

enqueue(nbr to visit_queue)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

end if

end for loop

end while loop

output ← parent, distance

state ← **start**

while state != **success** and state != **error**

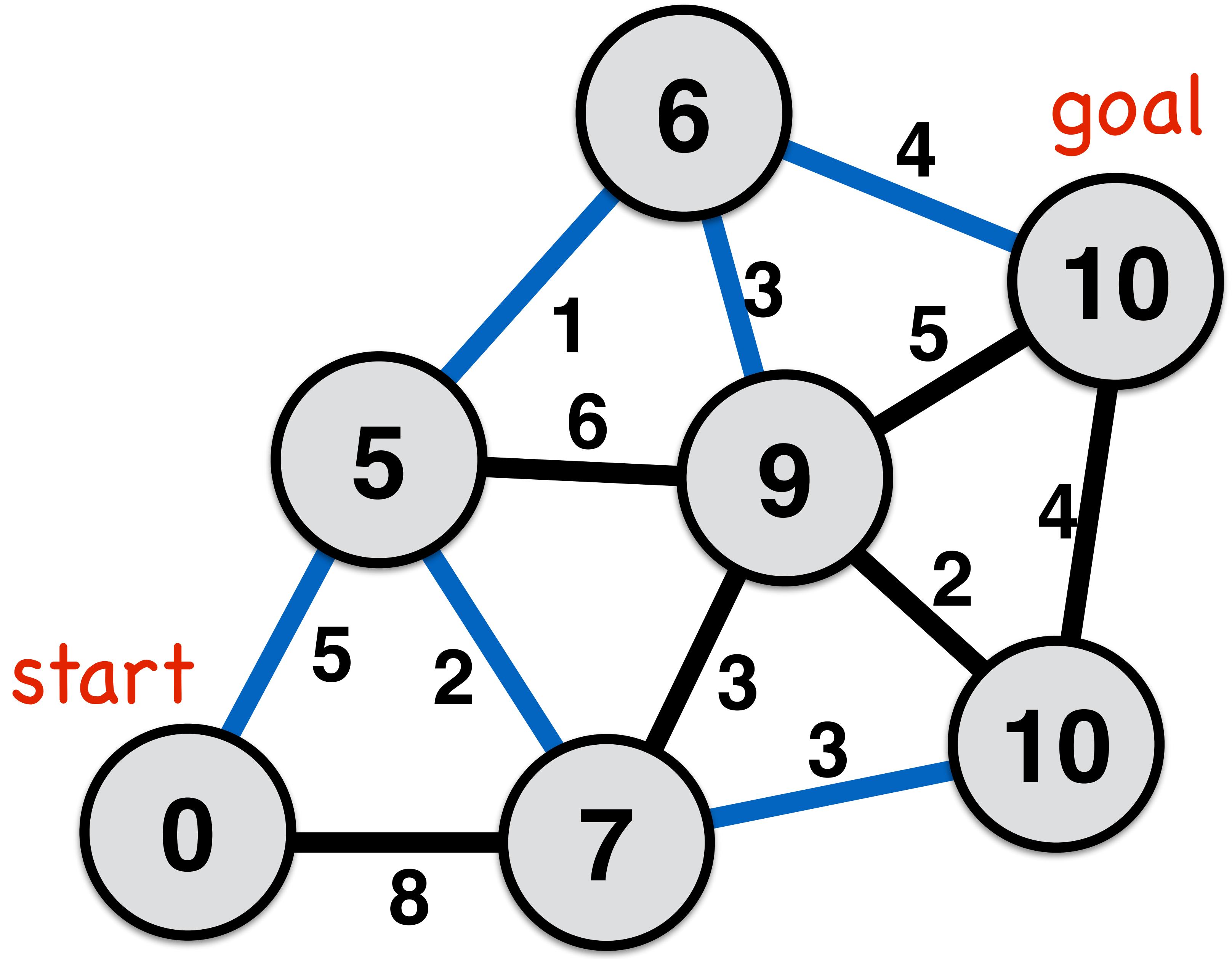
token ← next character

switch (state)

case **start**

if token = "n" then state ← **n_found**

else state ← **error**



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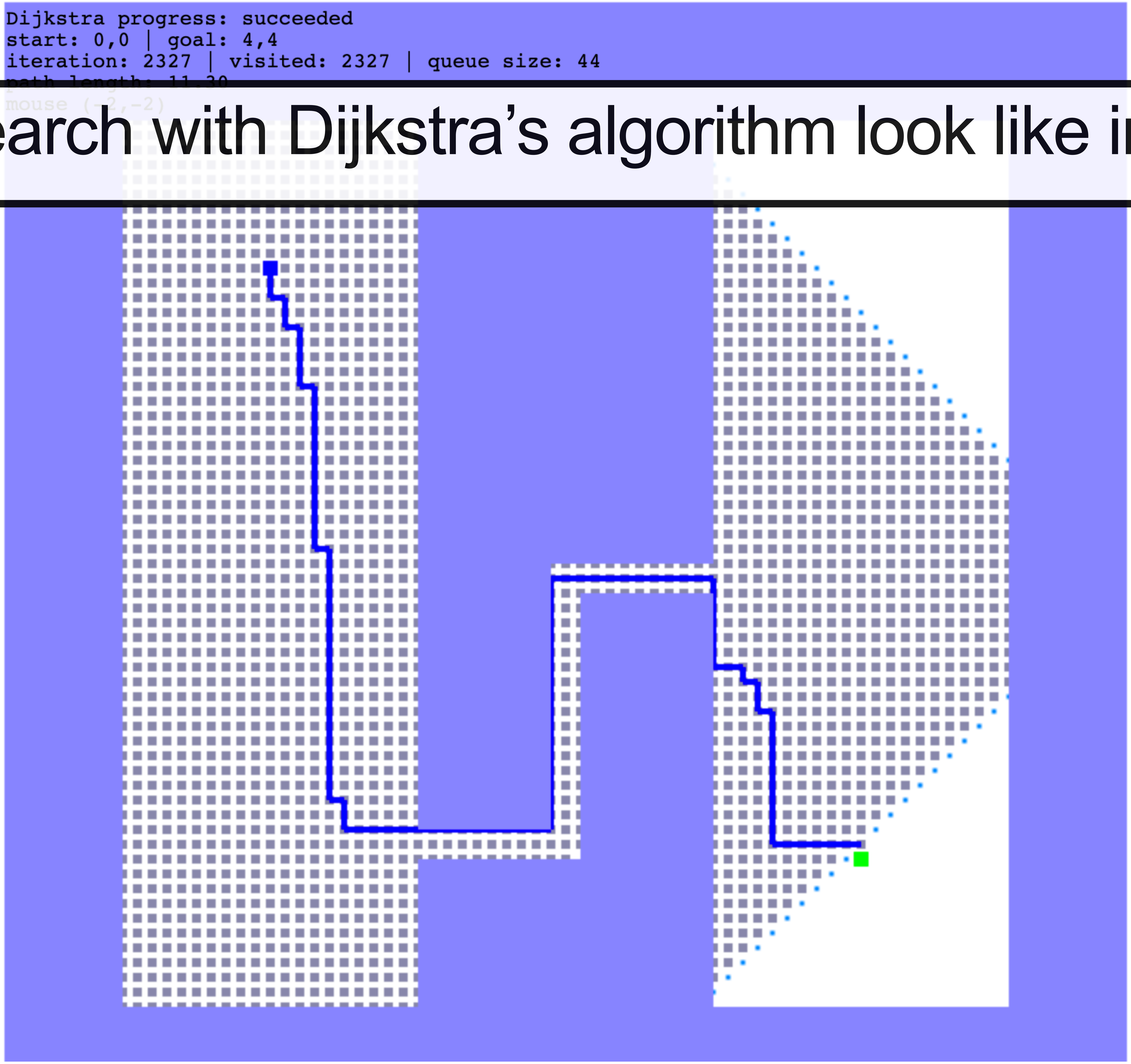
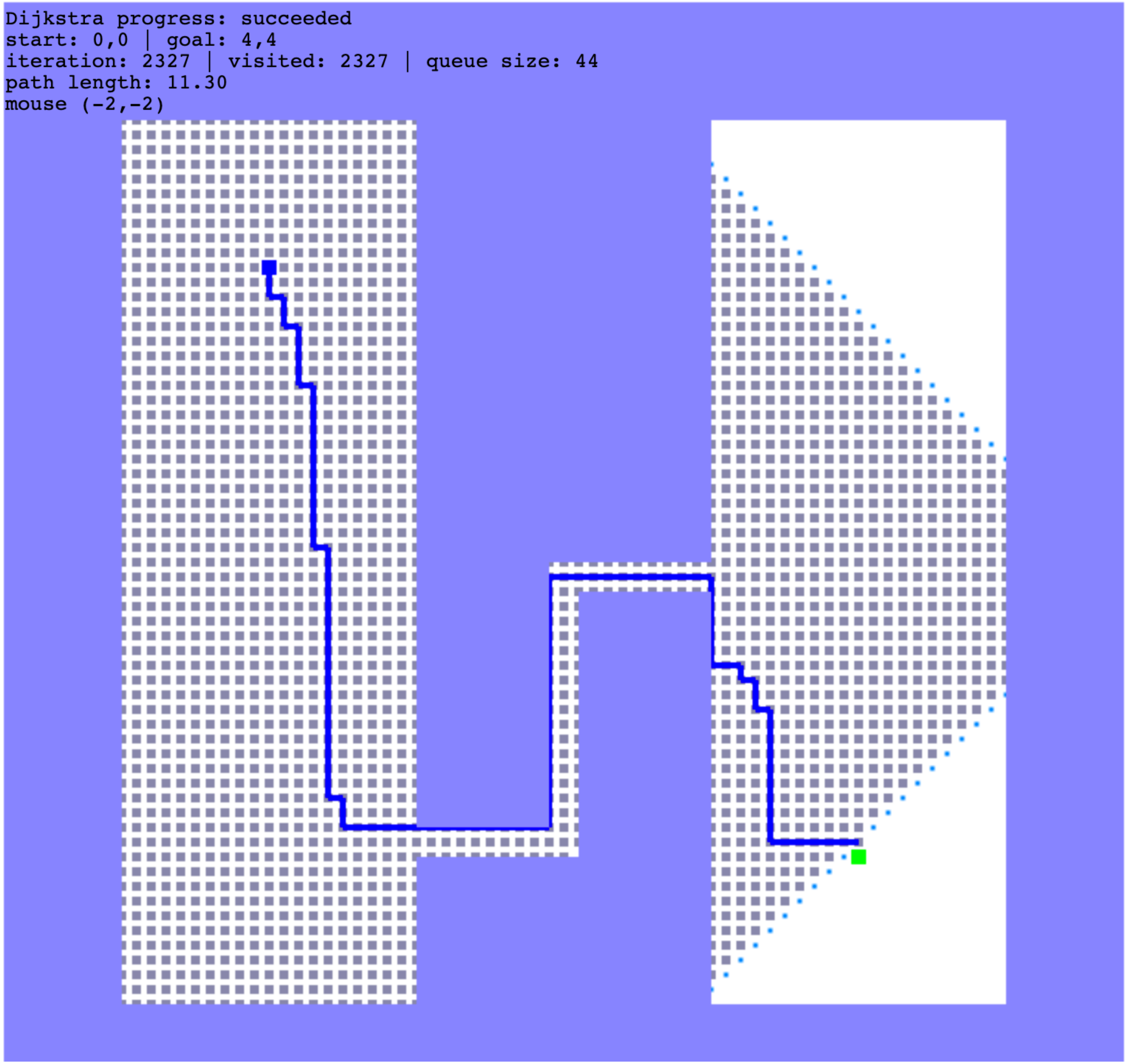
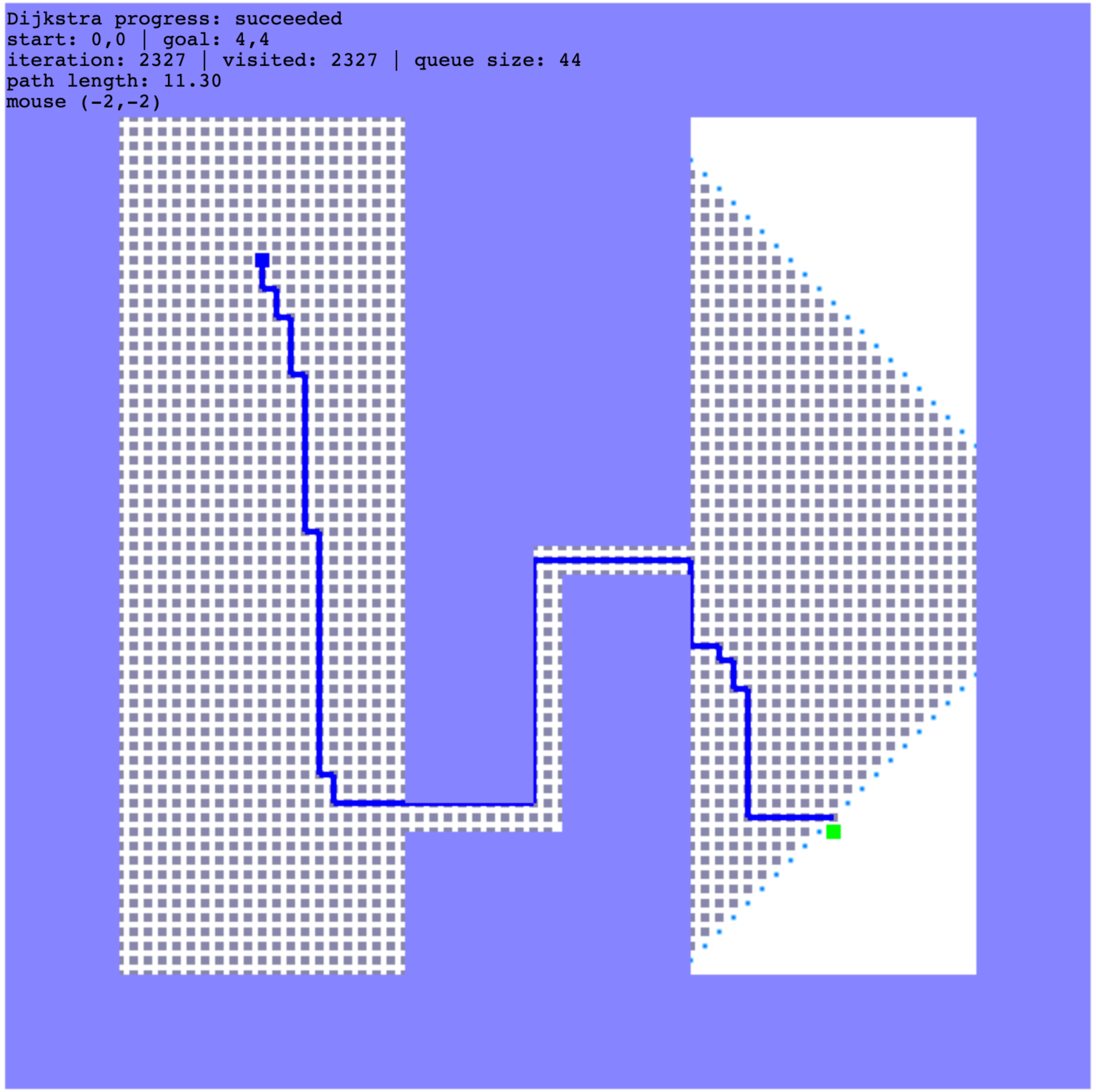


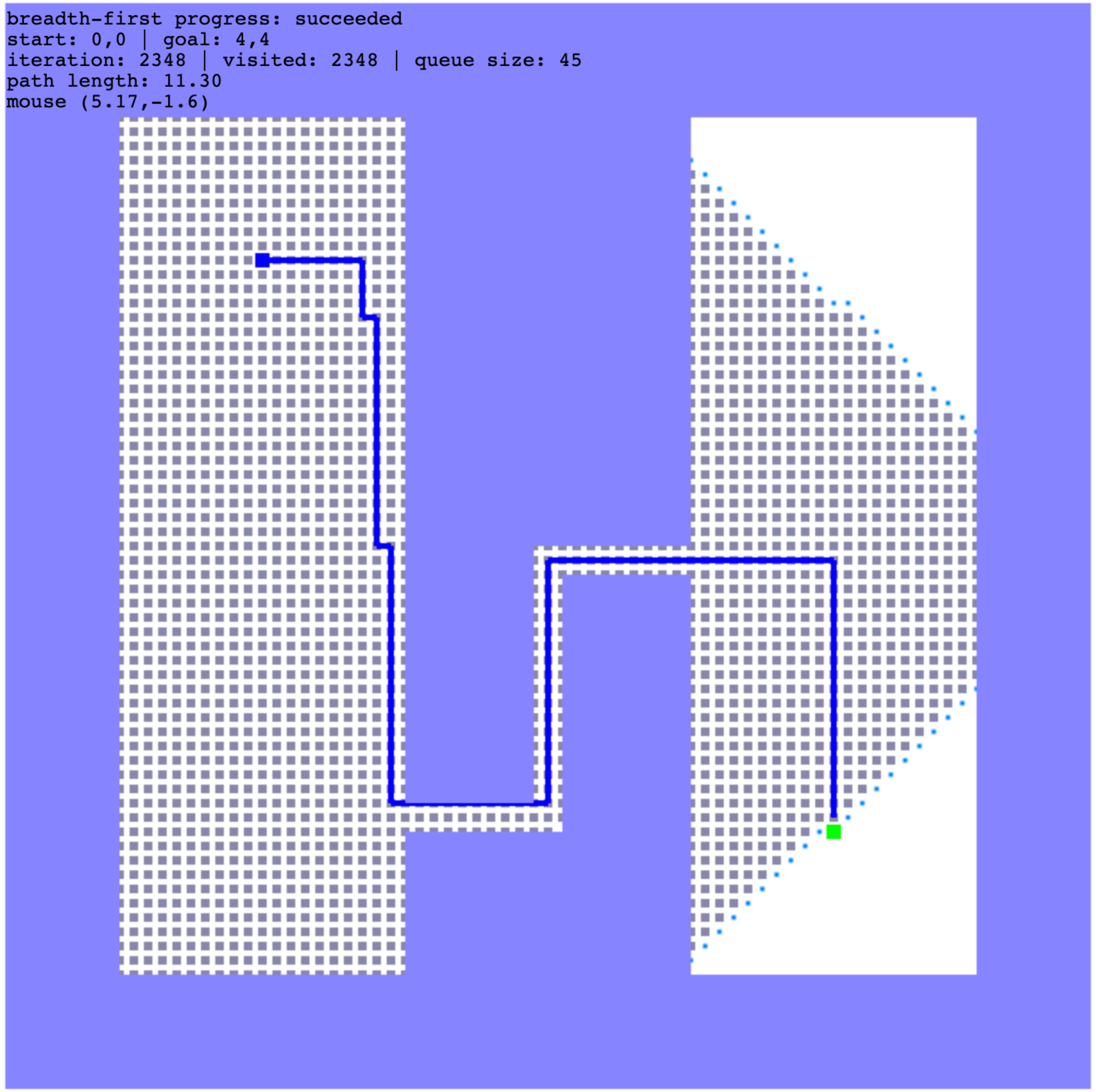
What will se… n this case?



Dijkstra



BFS



Why does their visit pattern look similar?

A-star Algorithm

A Formal Basis for the Heuristic Determination of Minimum Cost Paths

PETER E. HART, MEMBER, IEEE, NILS J. NILSSON, MEMBER, IEEE, AND BERTRAM RAPHAEL

Abstract—Although the problem of determining the minimum cost path through a graph arises naturally in a number of interesting applications, there has been no underlying theory to guide the development of efficient search procedures. Moreover, there is no adequate conceptual framework within which the various ad hoc search strategies proposed to date can be compared. This paper describes how heuristic information from the problem domain can be incorporated into a formal mathematical theory of graph searching and demonstrates an optimality property of a class of search strategies.

I. Introduction

A. *The Problem of Finding Paths Through Graphs*

MANY PROBLEMS of engineering and scientific importance can be related to the general problem of finding a path through a graph. Examples of such problems include routing of telephone traffic, navigation through a maze, layout of printed circuit boards, and mechanical theorem-proving and problem-solving. These problems have usually been approached in one of two ways, which we shall call the *mathematical approach* and the *heuristic approach.*

1) The mathematical approach typically deals with the properties of abstract graphs and with algorithms that prescribe an orderly examination of nodes of a graph to establish a minimum cost path. For example, Pollock and Wiebenson[1] review several algorithms which are guaranteed to find such a path for any graph. Busacker and Saaty[2] also discuss several algorithms, one of which uses the concept of dynamic programming.[3] The mathematical approach is generally more concerned with the ultimate achievement of solutions than it is with the computational feasibility of the algorithms developed.

2) The heuristic approach typically uses special knowledge about the domain of the problem being represented by a graph to improve the computational efficiency of solutions to particular graph-searching problems. For example, Gelernter's[4] program used Euclidean diagrams to direct the search for geometric proofs. Samuel[5] and others have used ad hoc characteristics of particular games to reduce

Manuscript received November 24, 1967.
The authors are with the Artificial Intelligence Group of the Applied Physics Laboratory, Stanford Research Institute, Menlo Park, Calif.

Hart, Nilsson, and Raphael

IEEE Transactions of System Science and Cybernetics, 4(2):100-107, 1968

Dijkstra shortest path algorithm

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_queue ← start_node

    while visit_queue != empty ~~&& current_node != goal~~

        cur_node ← min_distance(visit_queue)

        $visited_{cur\_node}$ ← true

        for each nbr in not_visited(adjacent(cur_node))

            enqueue(nbr to visit_queue)

            if $dist_{nbr} > dist_{cur\_node}$ + distance(nbr,cur_node)

                $parent_{nbr}$ ← current_node

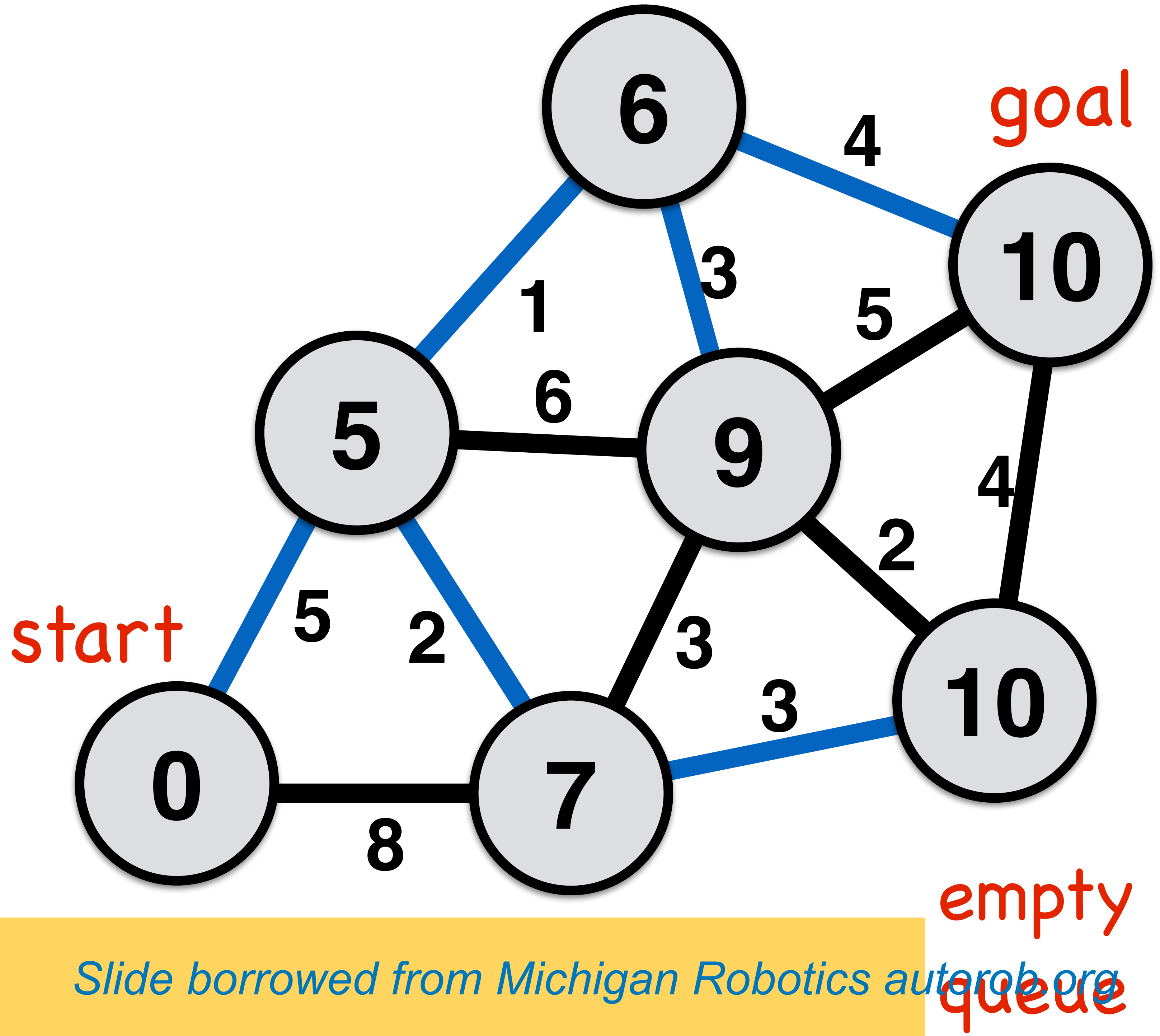
                $dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

            end if

        end for loop

    end while loop

output ← parent, distance



A-star shortest path algorithm

all nodes ← {$dist_{start}$ ← infinity, $parent_{start}$ ← none, $visited_{start}$ ← false}

start_node ← {$dist_{start}$ ← 0, $parent_{start}$ ← none, $visited_{start}$ ← true}

visit_queue ← start_node

while (visit_queue != empty) && current_node != goal

cur_node ← **dequeue(visit_queue, f_score)**

$visited_{cur\_node}$ ← true

for each nbr in not_visited(adjacent(cur_node))

enqueue(nbr to visit_queue)

if $dist_{nbr}$ > $dist_{cur\_node}$ + distance(nbr,cur_node)

$parent_{nbr}$ ← current_node

$dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)

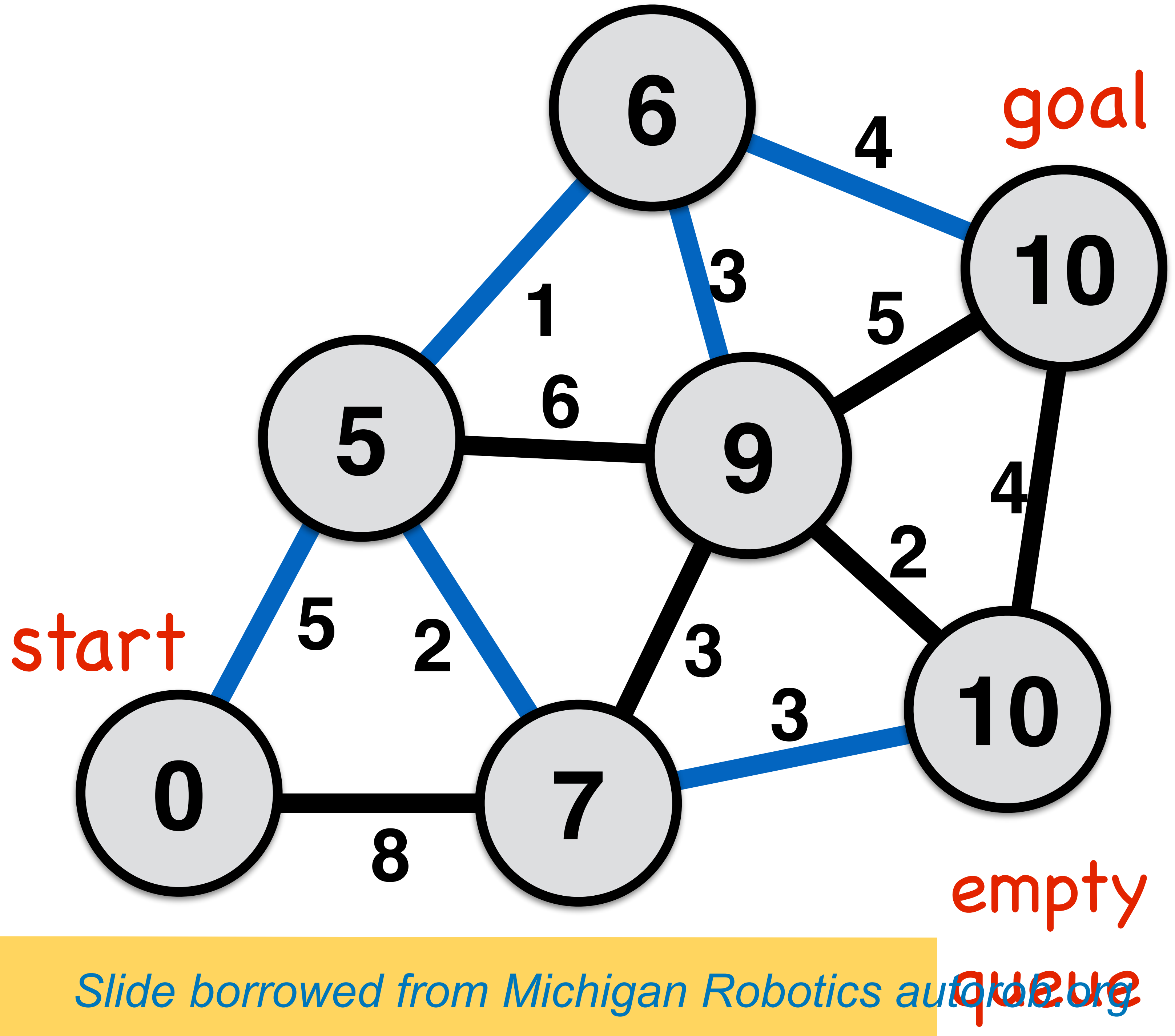
f_score ← $distance_{nbr}$ + $line\_distance_{nbr,goal}$

end if

end for loop

end while loop

output ← parent, distance



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                f_score ← $distance_{nbr}$ + $line\_distance_{nbr,goal}$

            end if

        end for loop

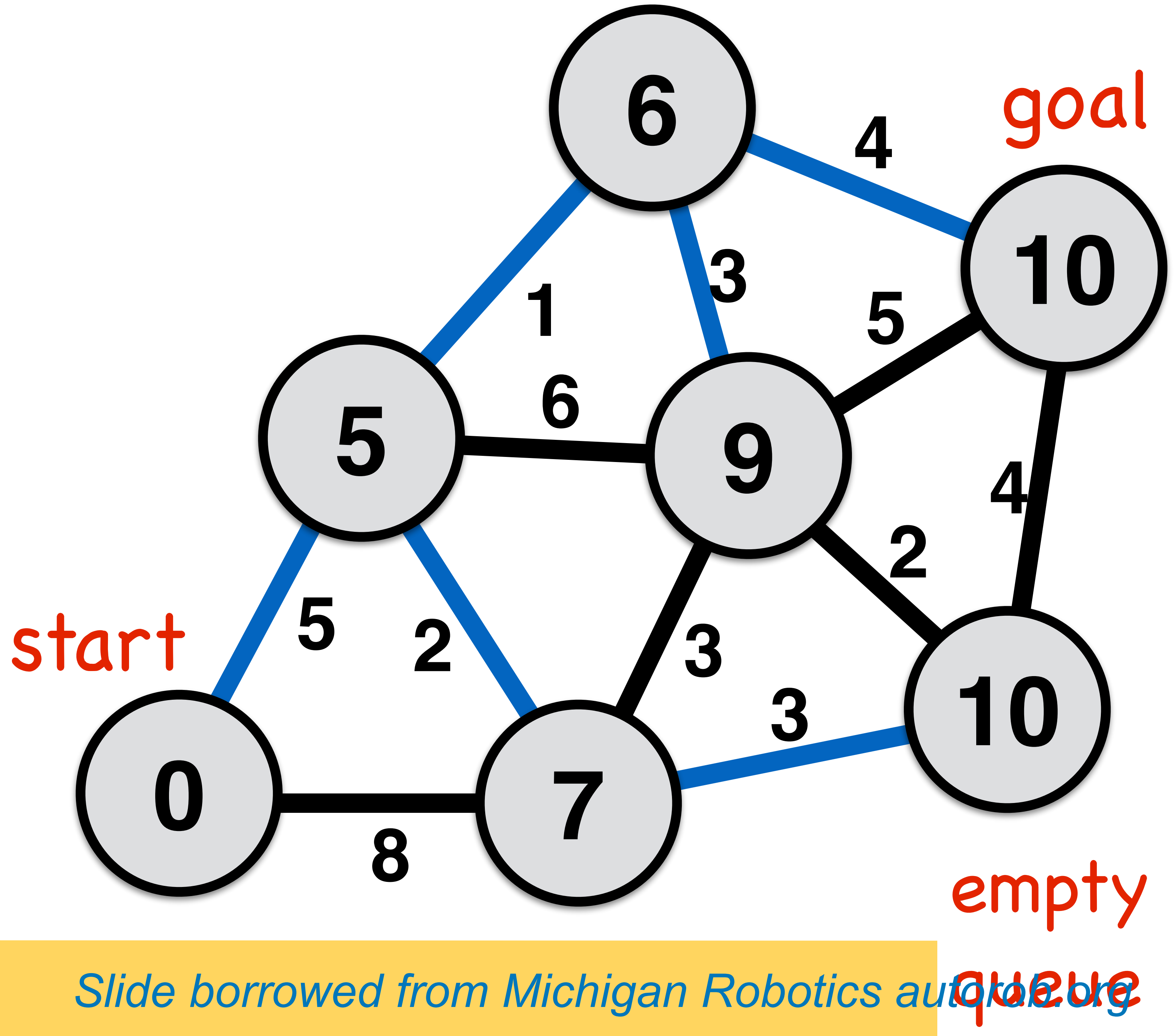
    end while loop

output ← parent, distance

priority queue wrt. f_score (implement min binary heap)

g_score: distance along current path back to start

h_score: best possible distance to goal



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Why is A-star advantageous?

visit_queue ← start_node

while (visit_queue != empty) && current_node != goal
- cur_node ← **dequeue(visit_queue, f_score)**
- $visited_{cur\_node}$ ← true
- **for** each nbr in not_visited(adjacent(cur_node))
  - enqueue(nbr to visit_queue)
  - **if** $dist_{nbr} > dist_{cur\_node}$ + distance(nbr,cur_node)
    - $parent_{nbr}$ ← current_node
    - $dist_{nbr}$ ← $dist_{cur\_node}$ + distance(nbr,cur_node)
    - **f_score ← $distance_{nbr}$ + $line\_distance_{nbr,goal}$**
  - **end** if
- **end** for loop

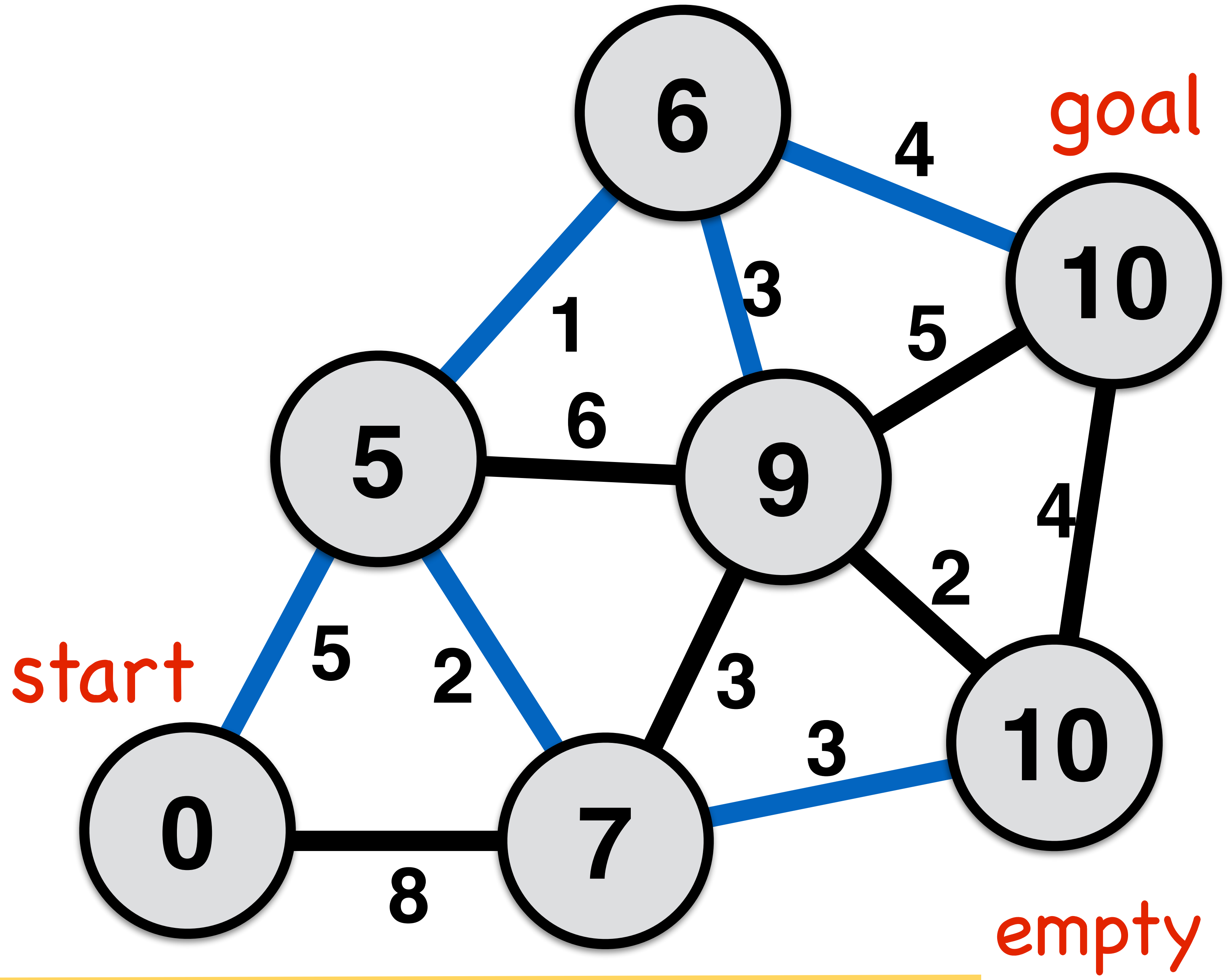
end while loop

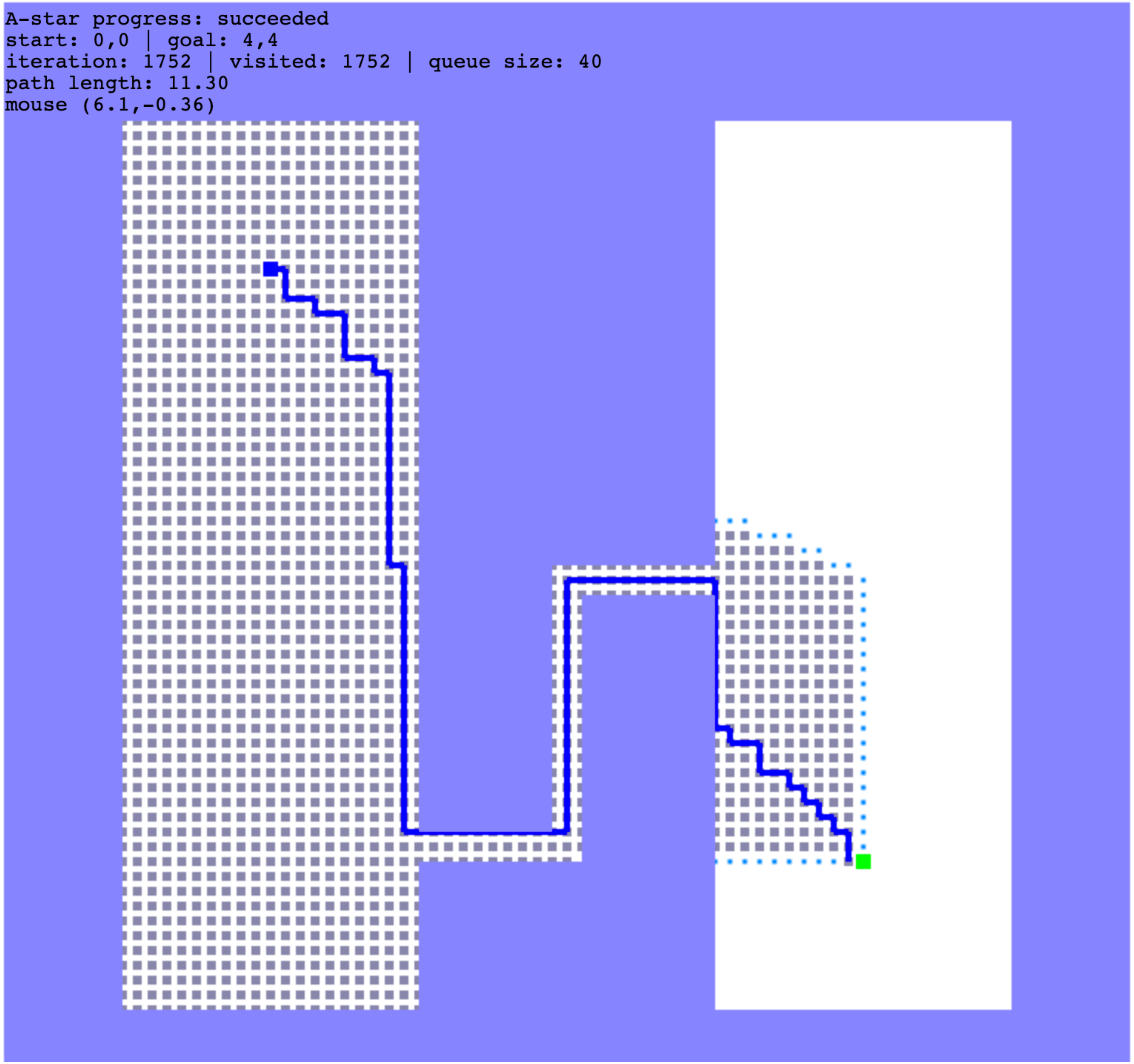
output ← parent, distance

priority queue wrt. f_score (implement min binary heap)

g_score: distance along current path back to start

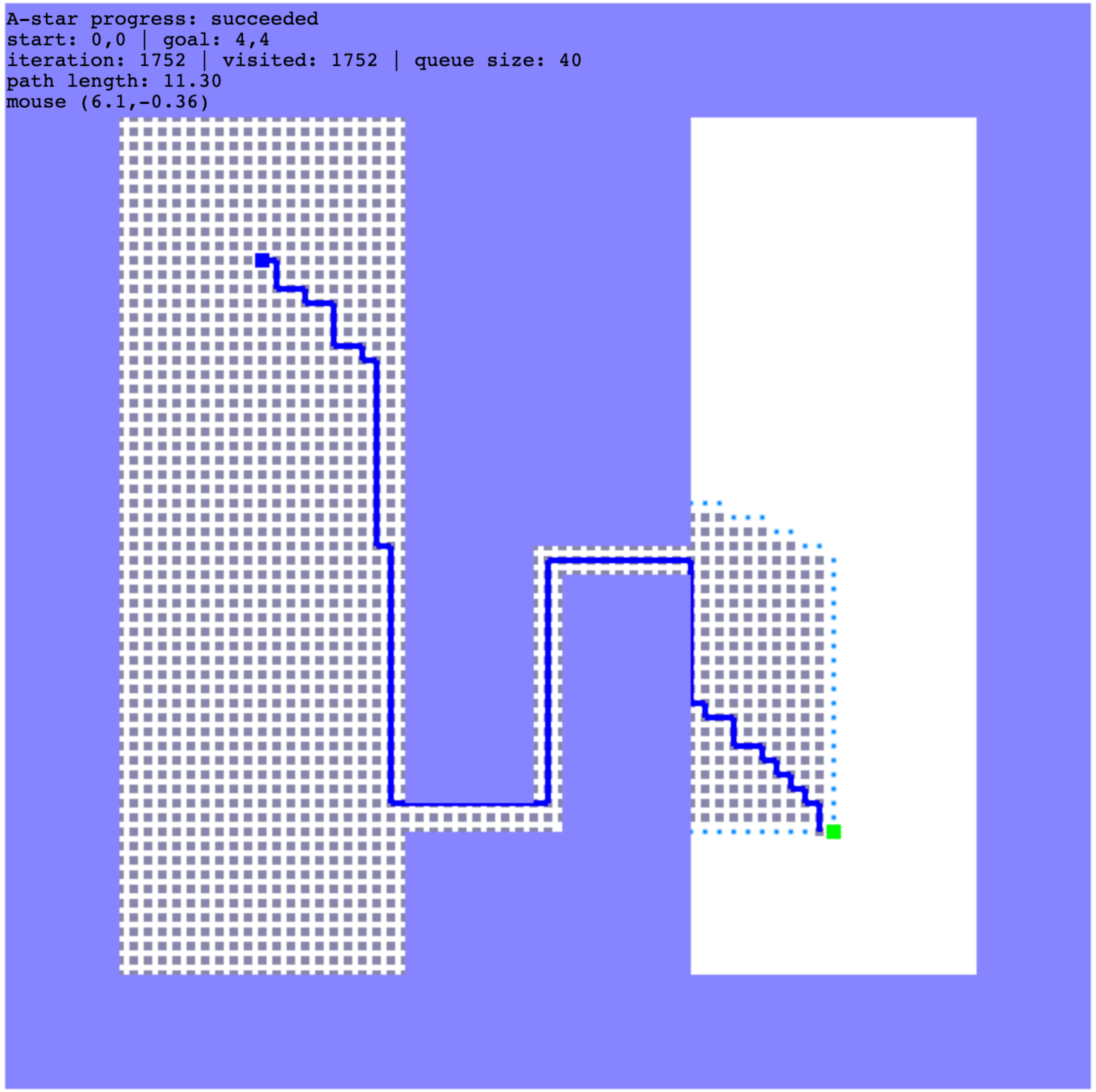
h_score: best possible distance to goal



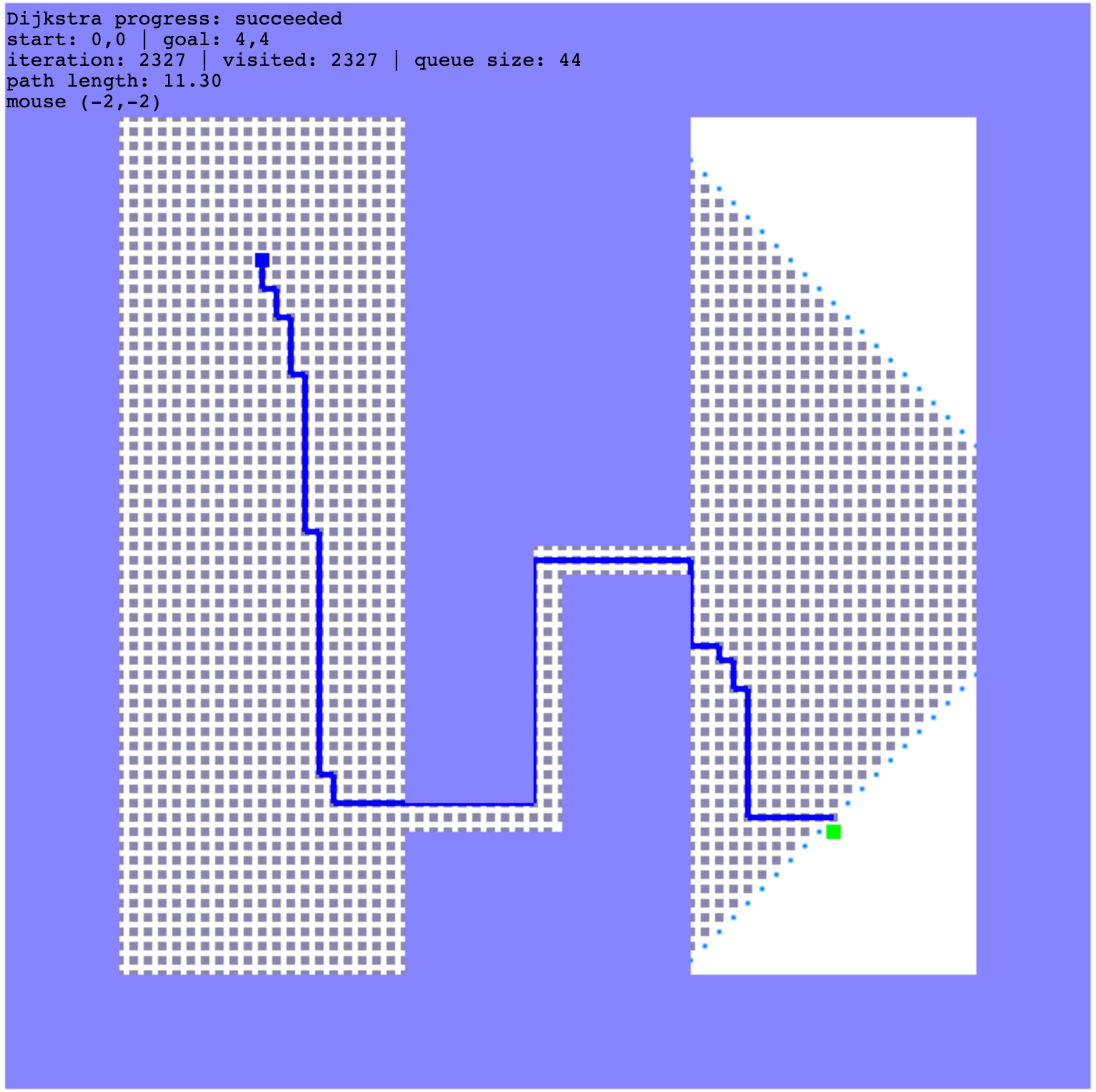


Slide borrowed from Michigan Robotics autorob.org

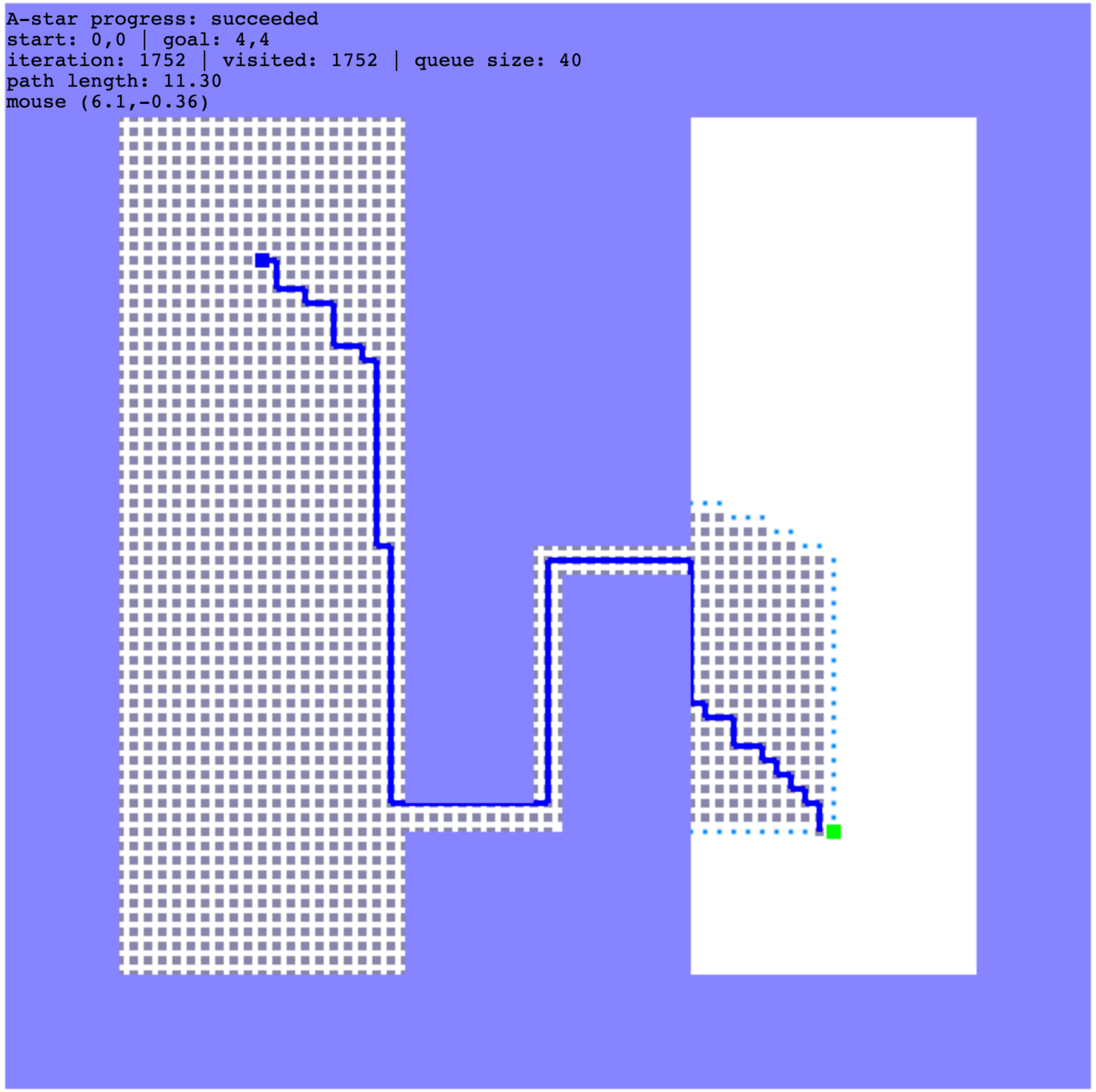
A-Star



Dijkstra

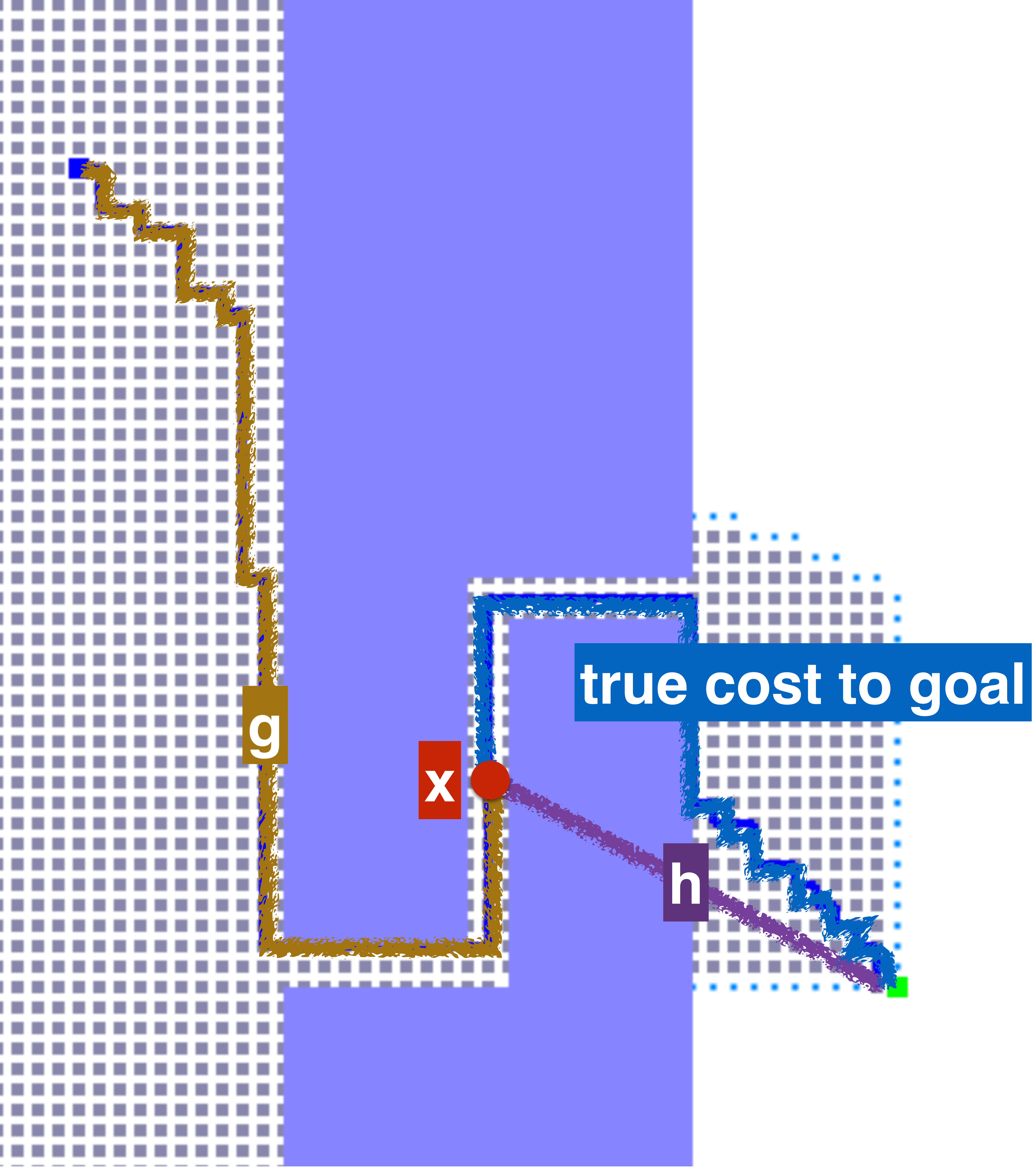


How can A-star visit few nodes?



How can A-star visit few nodes?

A-Star uses an admissible heuristic to estimate the cost to goal from a node



The straight line h_score is an admissible and consistent heuristic function.

A heuristic function is **admissible** if it never overestimates the cost of reaching the goal.

> Thus, h_score(x) is less than or equal to the lowest possible cost from current location to the goal.

A heuristic function is **consistent** if obeys the triangle inequality

> Thus, h_score(x) is less than or equal to cost(x,action,x') + h_score(x')

https://www.cs.cmu.edu/~./awm/tutorials/astar08.pdf

Proof: A* with Admissible Heuristic Guarantees Optimal Path

- Suppose it finds a suboptimal path, ending in goal state G_1 where $f(G_1) > f^*$ where $f^* = h^*(start)$ = cost of optimal path.
- There must exist a node n which is
  - Unexpanded
  - The path from start to n (stored in the BackPointers(n) values) is the start of a true optimal path

Why must such a node exist? Consider any optimal path $s, n1, n2 \ldots$goal. If all along it were expanded, the goal would've been reached along the shortest path.

- $f(n) >= f(G_1)$ (else search wouldn't have ended)
- Also $f(n) = g(n) + h(n)$

  $= g^*(n) + h(n)$ — because it's on optimal path

  $<= g^*(n) + h^*(n)$ — By the admissibility assumption

  $= f^*$ — Because n is on the optimal path

So $f^* >= f(n) >= f(G_1)$ — contradicting top of slide

Slide 21

Next Lecture
Linear Algebra Refresher