# Lecture 03

# Linear Algebra Refresher

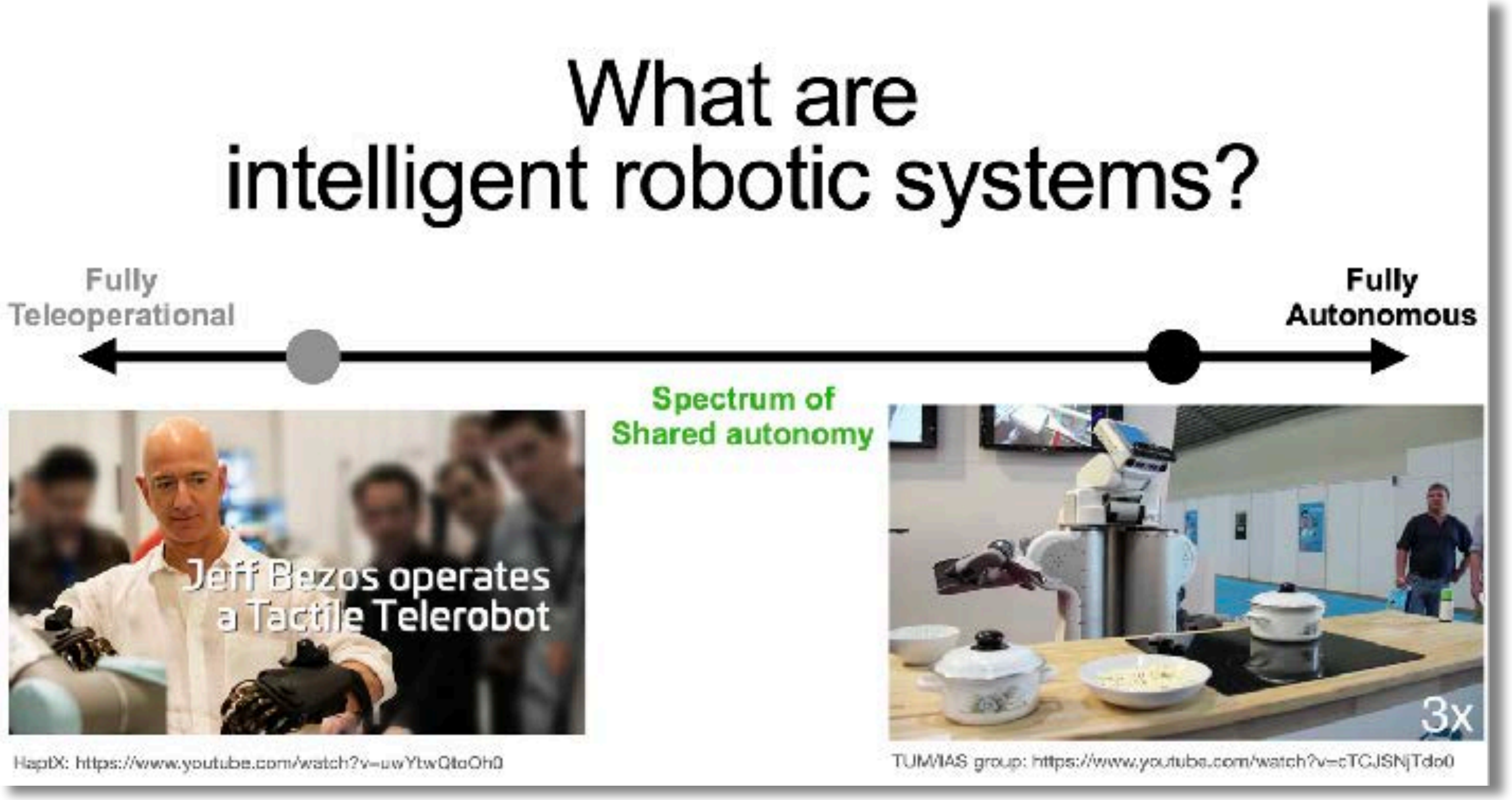
# Course Logistics

- OH Starting this week! Details on the course webpage.
- Everyone should be on Ed discussion board now.
- Everyone should be on Gradescope now.
- Quiz 1 was posted yesterday and was due today at **noon**.
- Project 1 will be posted today 01/24 and will be due 01/31.
- **Action items for you:**
  - Announcements are done via Ed discussion. So make sure you are getting email notifications from Ed.

# FAQs

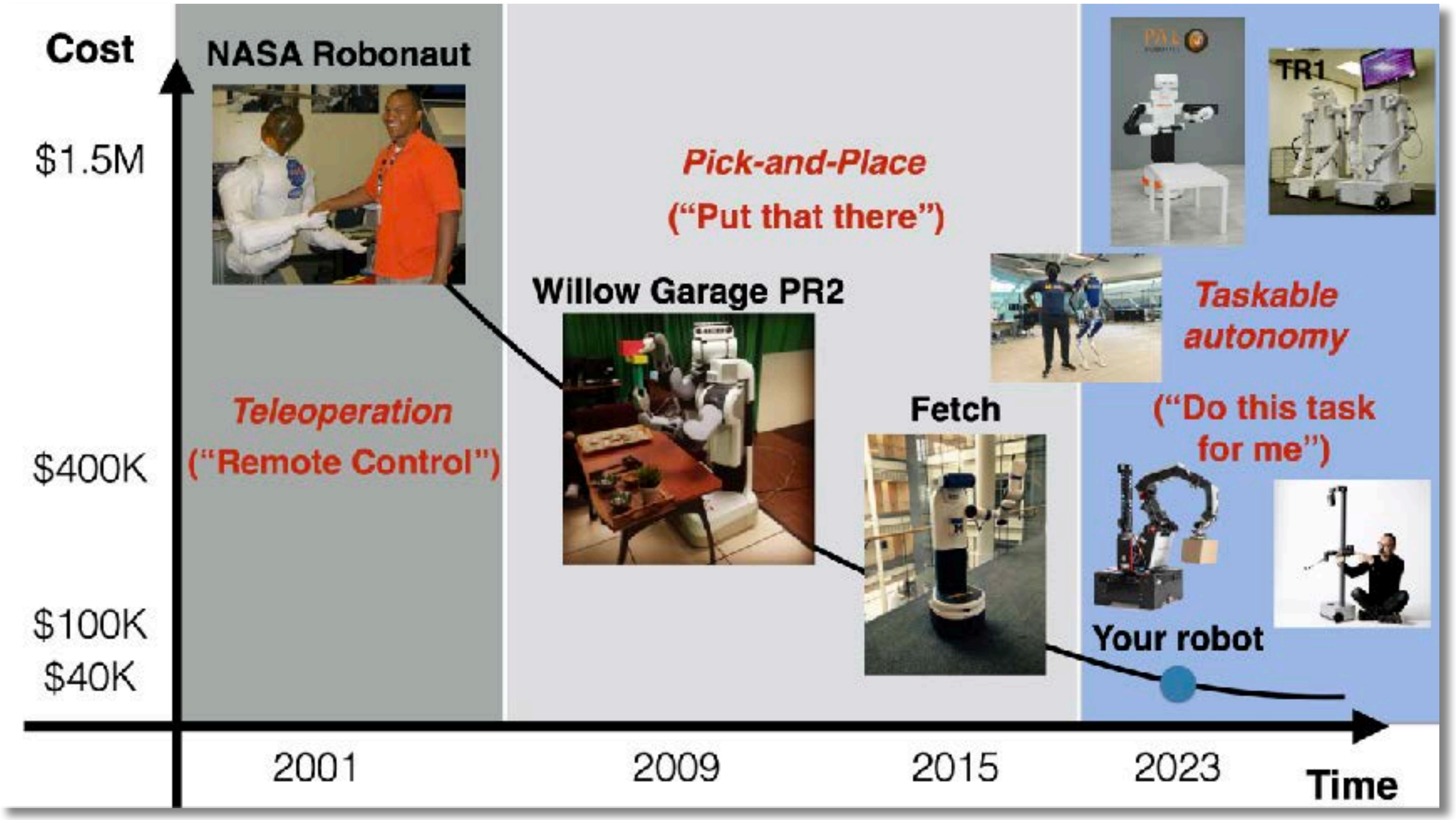
- **When will the slides be posted?**
  - Slides for the lecture will be posted on the course webpage by before every lecture.
- **How much programming knowledge is required for this course?**
  - You will use JavaScript for the most part.
  - Proficiency in any programming language (Python, C, C++, Java, JavaScript…) is essential to perform in the course.
  - Being able to debug your program effectively will be the key.
- **Do I need to know ROS?**
  - No!
- **Will I learn ROS?**
  - You will be exposed to ROS in P7.
- **Any tips to succeed in this course?**
  - Start the projects as and when they are released.
  - Use office hours and Ed discussion board.

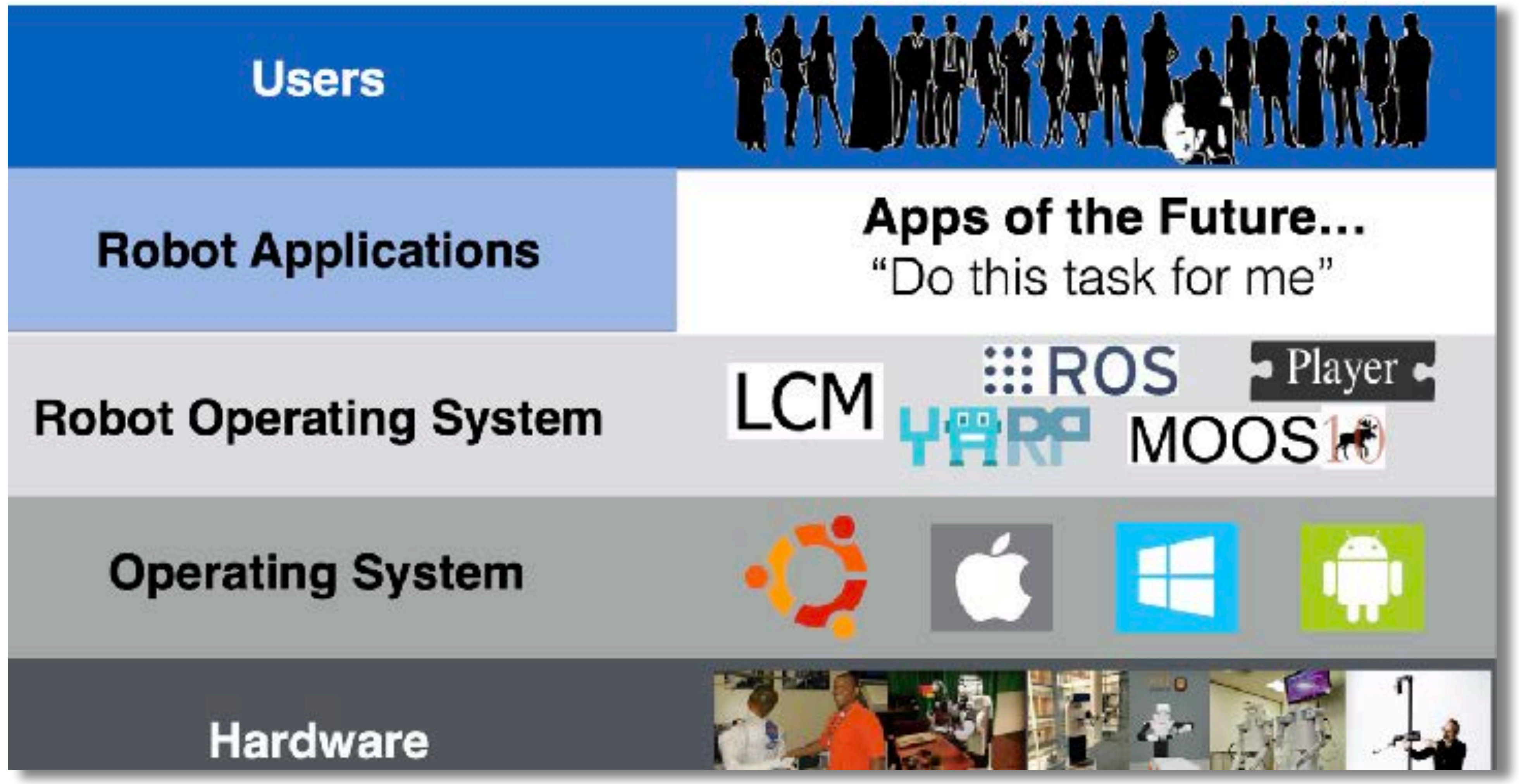
# Previously

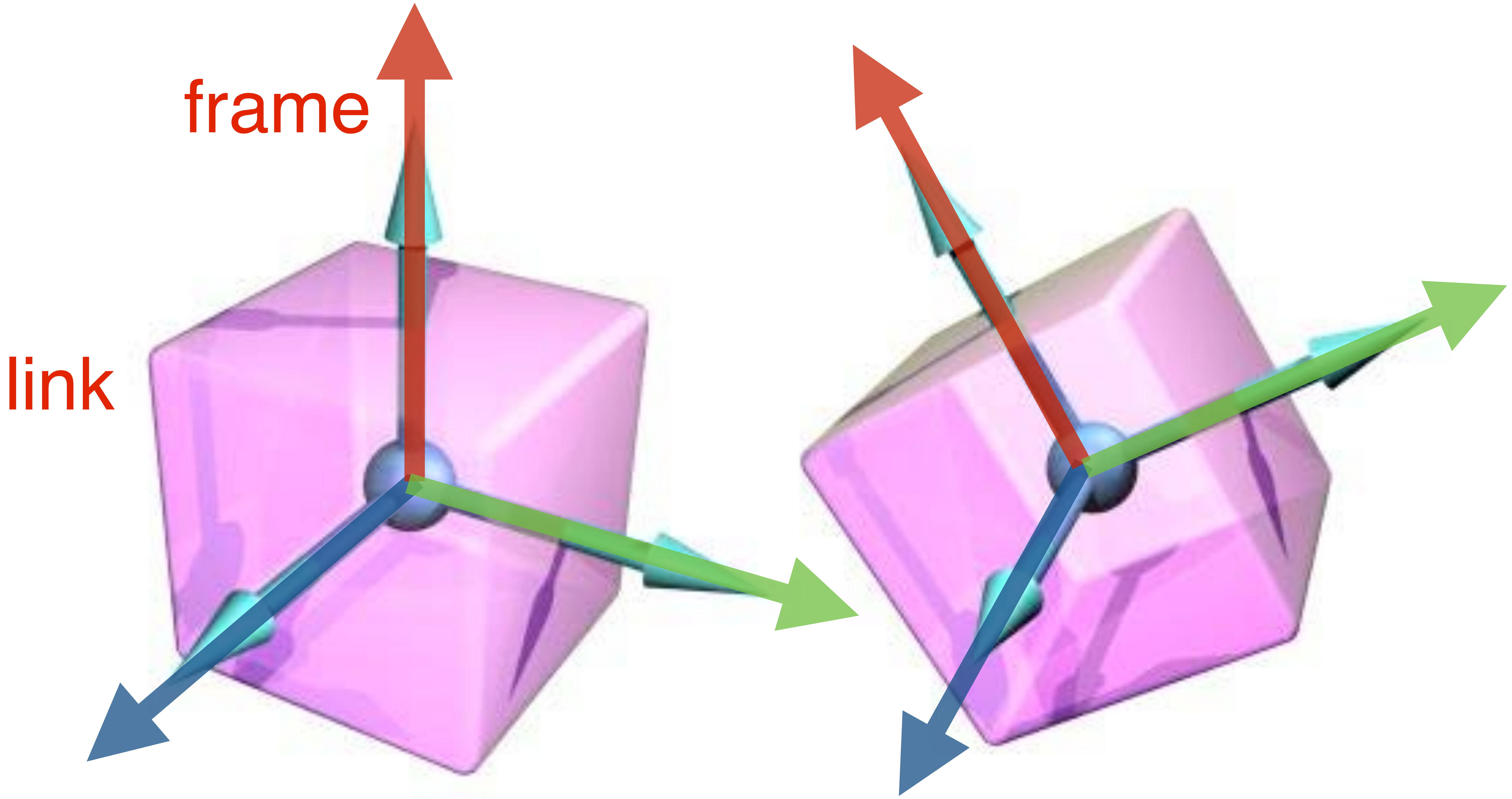



...systems that can perform Sense-Plan-Act....

... can also learn skills ... transfer these skills ... adapt to new environments ...

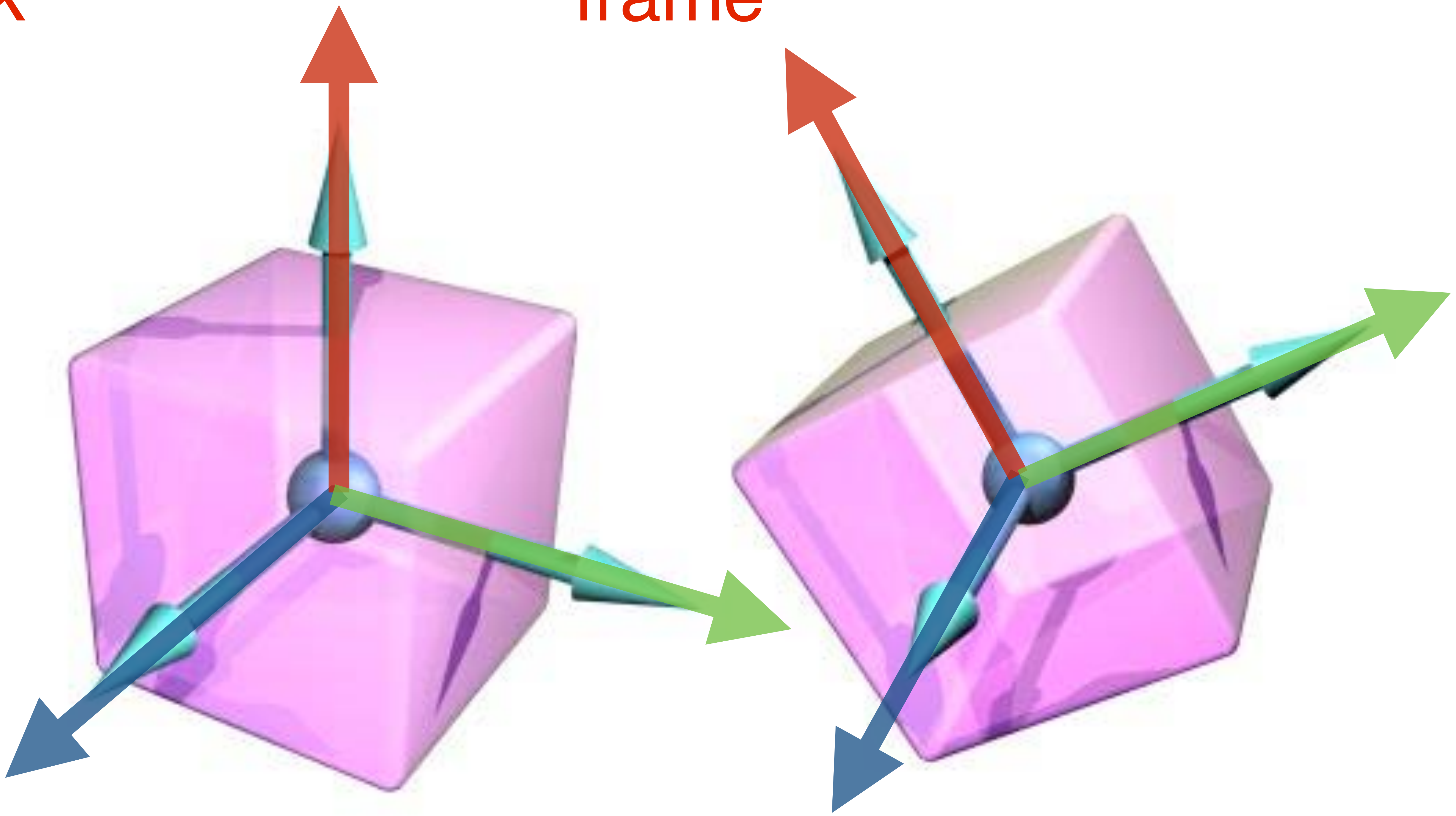

frame
link

# DOFs

- Each body has its own coordinate system

link

frame

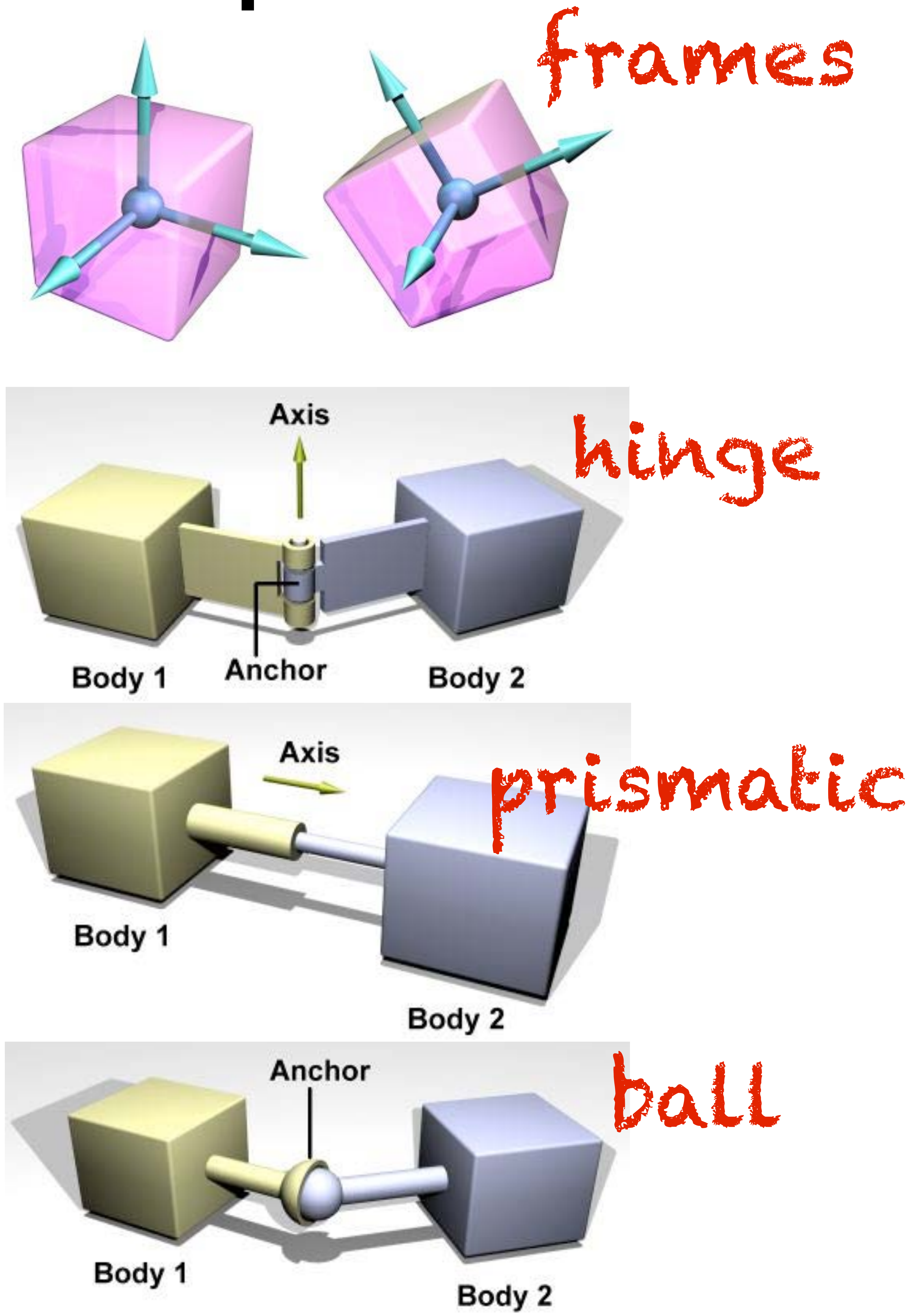


# Reset: DOFs and Coordinate Spaces

- Each <u>body</u> has its own frame
- Joints connect two links (rigid bodies)
  - e.g., hinge, prismatic, ball-socket
- A motor exerts force on a DOF axis

Linear algebra

- Matrix transformations used to relate coordinate systems of bodies and joints
- Spatial geometry attached to each link, but does not affect the body's coordinate frame



# Reset: DOFs and Coordinate Spaces

- Each body has its own frame

**Rigid Body**

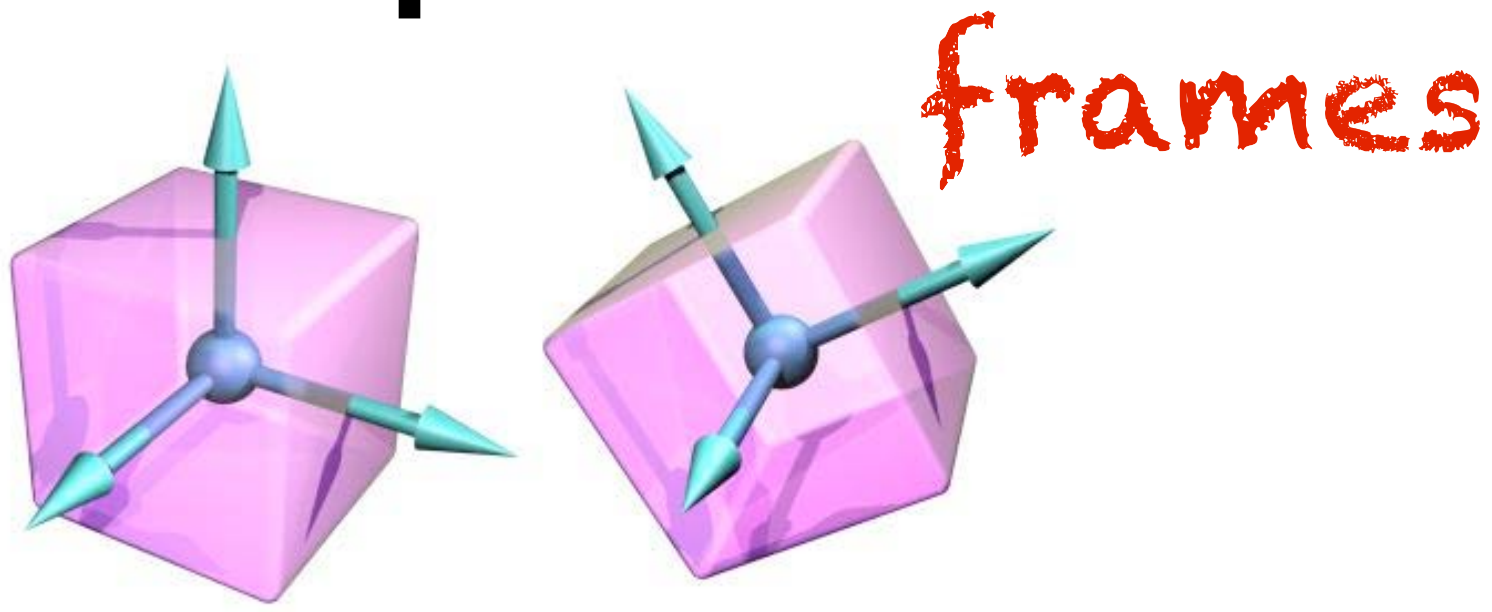
**vs.**

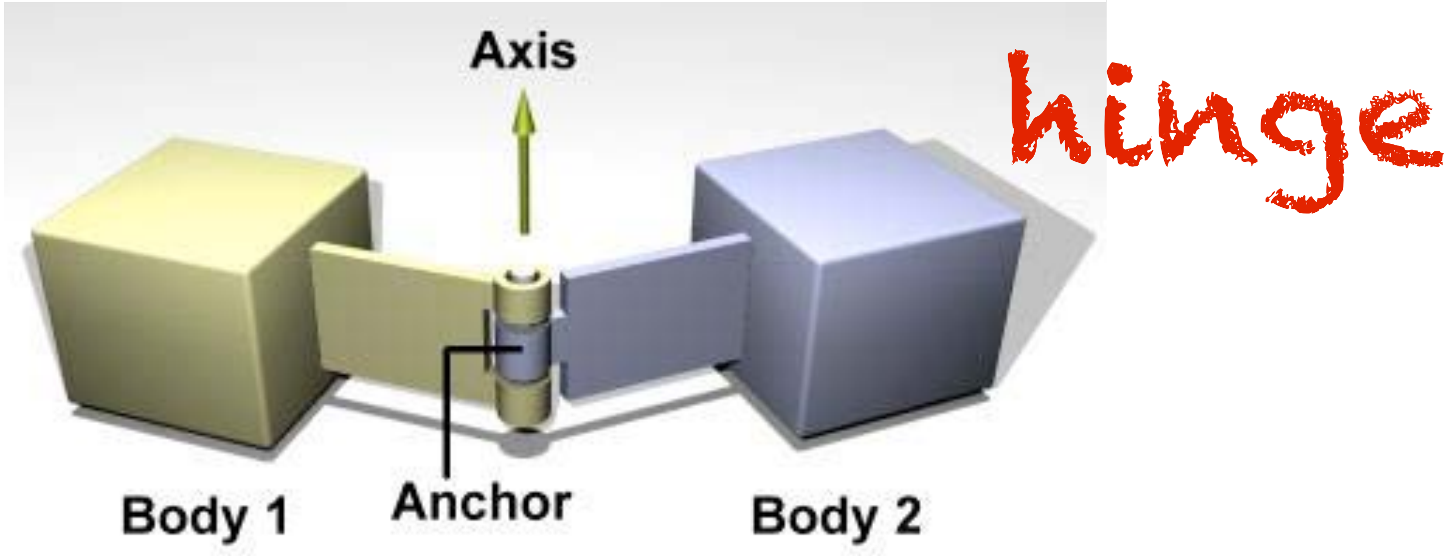
**Link**

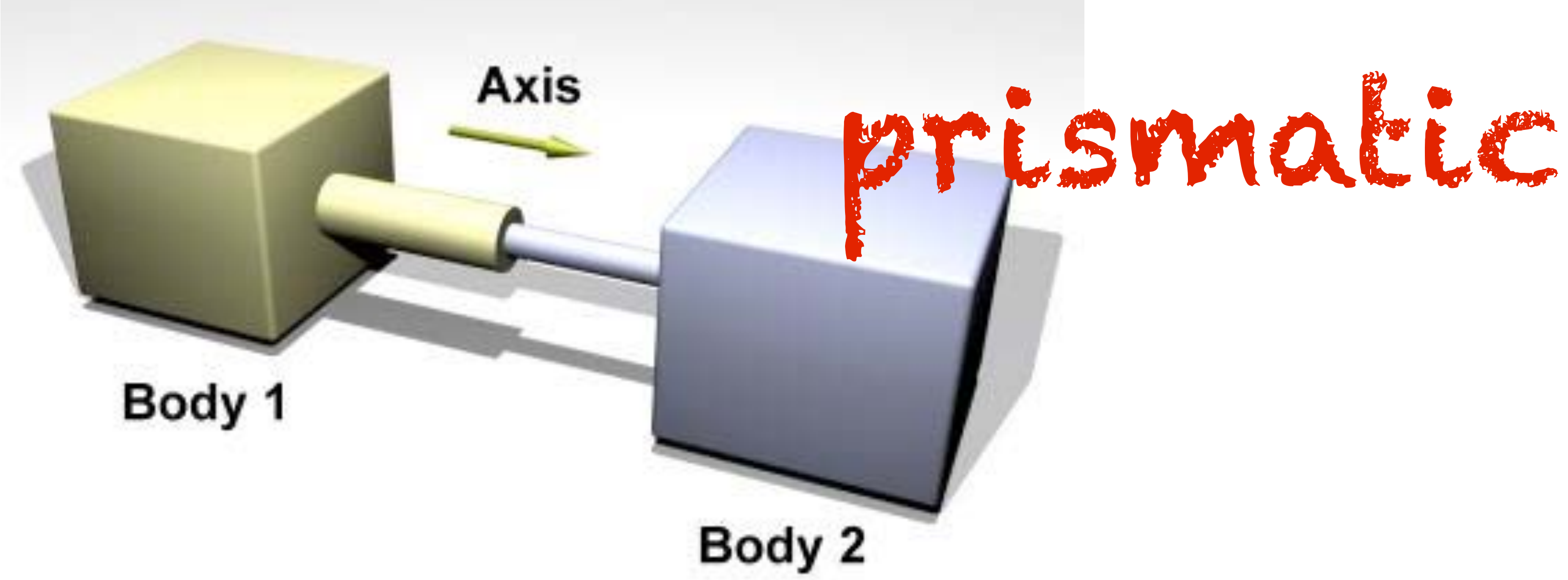
**vs.**

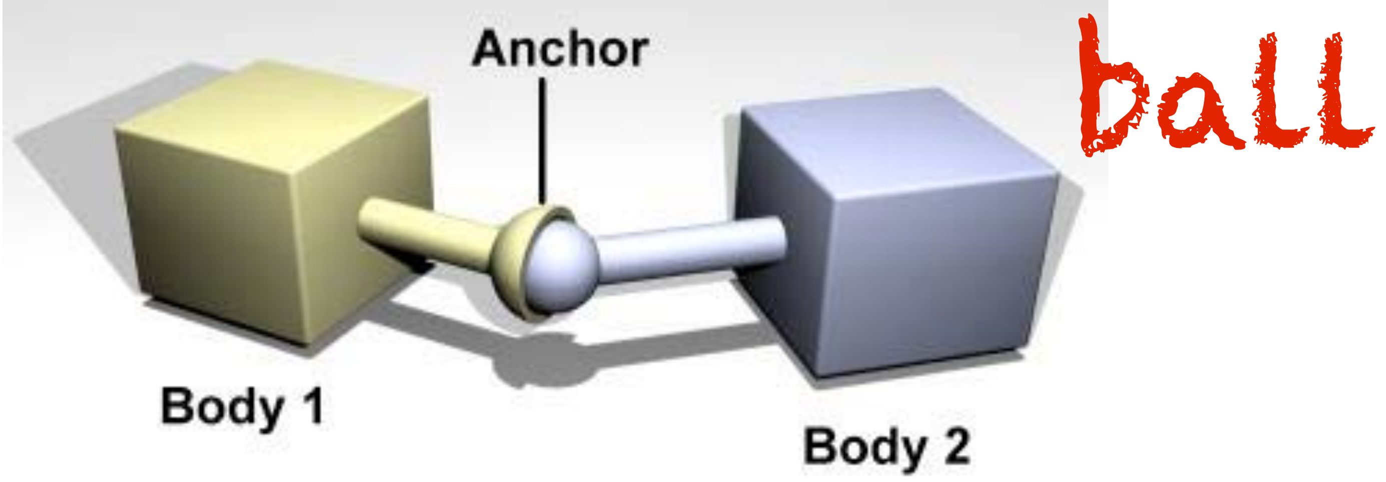
**Joint**

- Spatial geometry attached to each link, but does not affect the body's coordinate frame

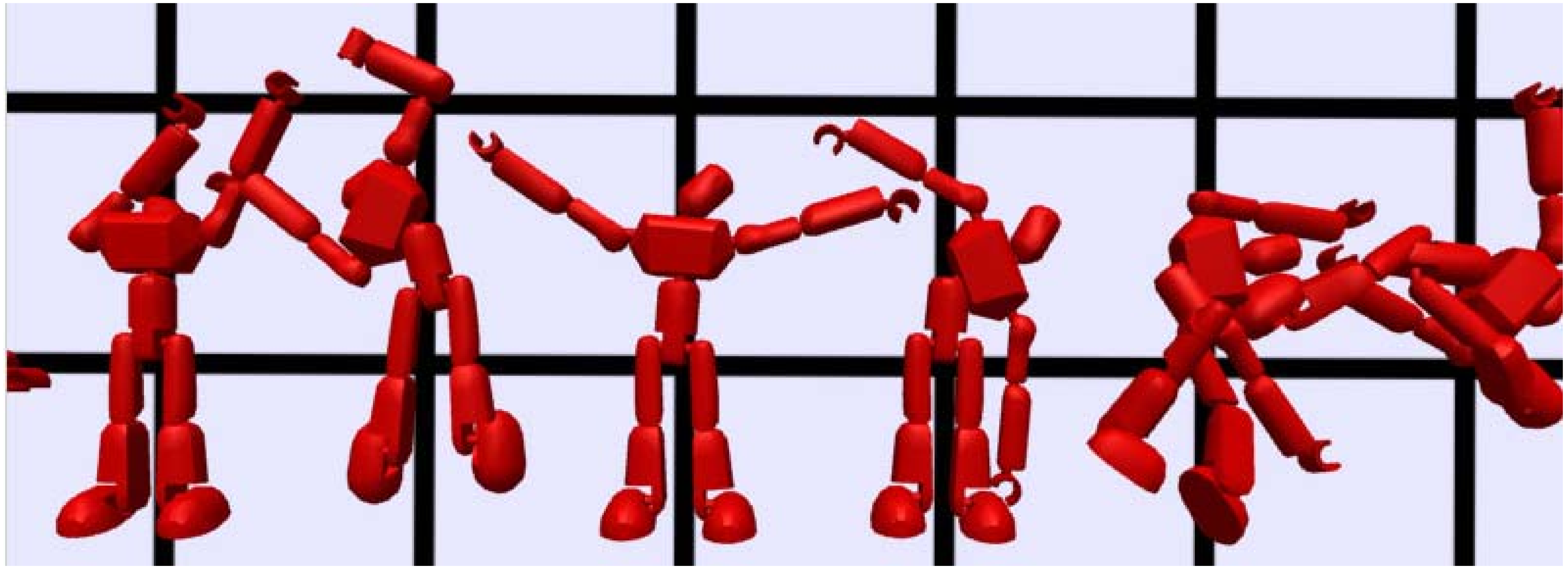


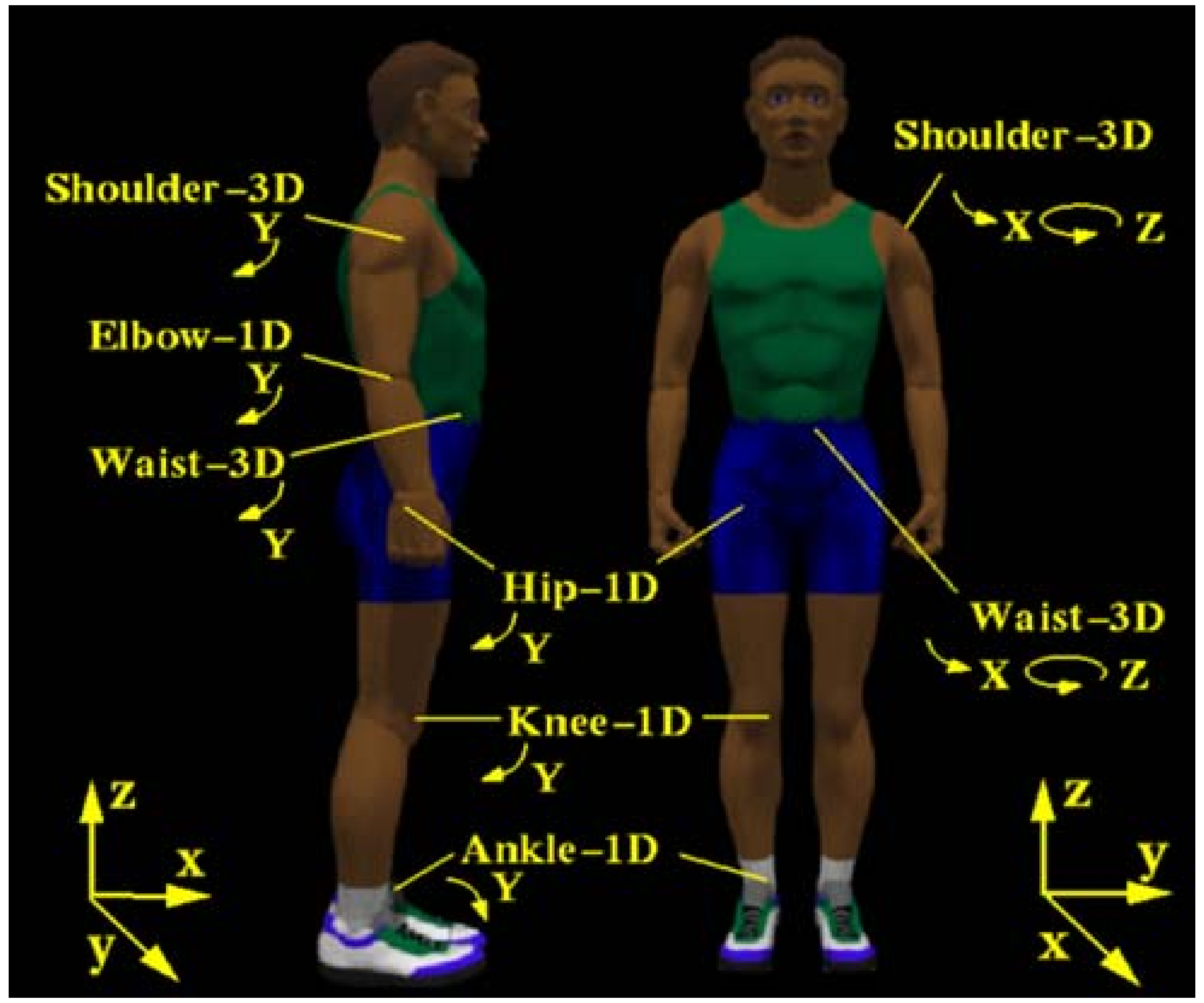



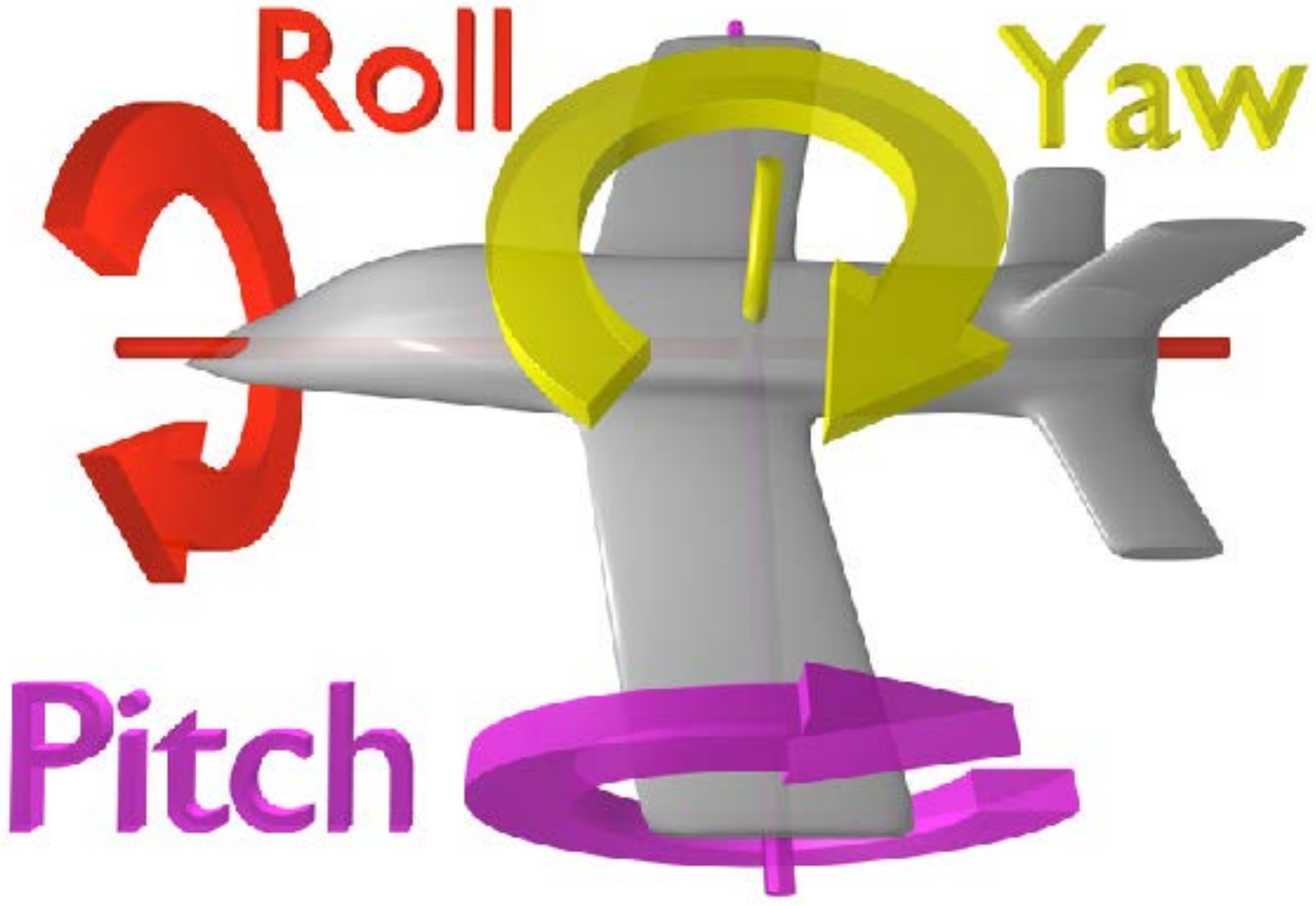



# Reset: Kinematics



- State comprised of degrees-of-freedom (DOFs)
- DOFs describe translation and rotation axes of system



# DOF definitions

- Degrees of freedom of a rigid body: $\sum$ (freedoms of points) – # of independent constraints
- Degrees of freedom of a robot: $\sum$ (freedoms of bodies) – # of independent constraints

*We can use Grübler's formula to determine the DOF of a robot.*
*We will talk about this in the future classes.*

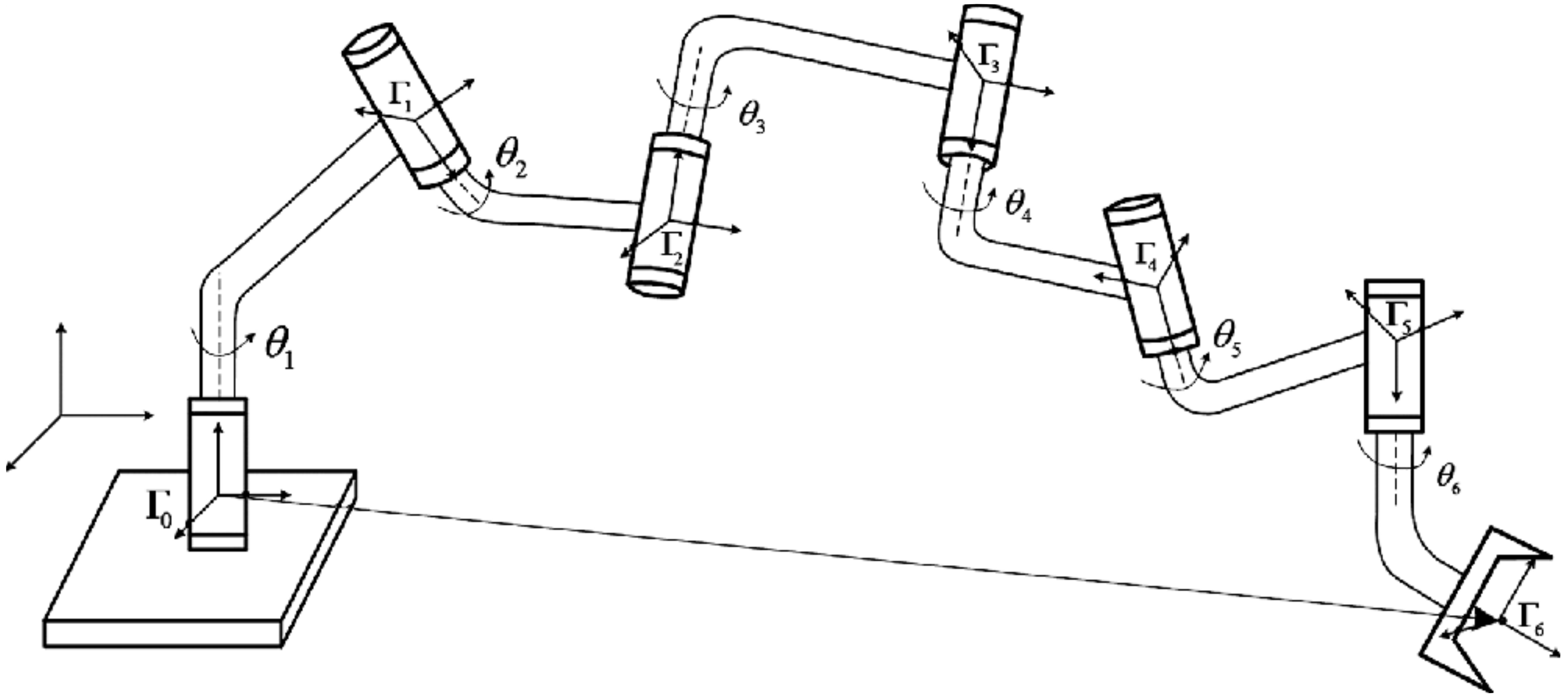
# Robot Kinematics

**Goal**: Given the structure of a robot arm, compute

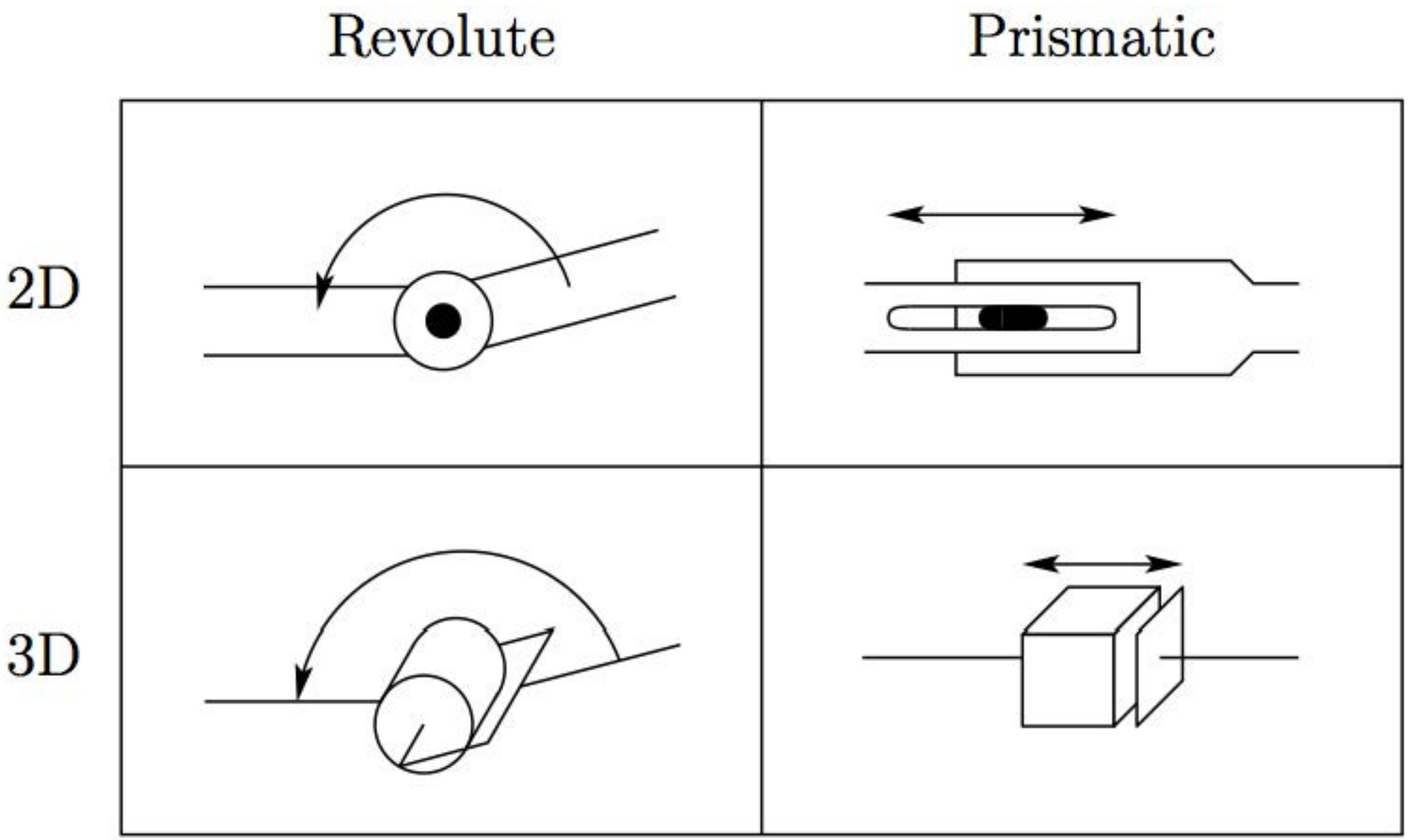
– **Forward kinematics**: inferring the pose of the end-effector, given the state of each joint.

– **Inverse kinematics**: inferring the joint states necessary to reach a desired end-effector pose.

But, we need to start with a linear algebra refresher



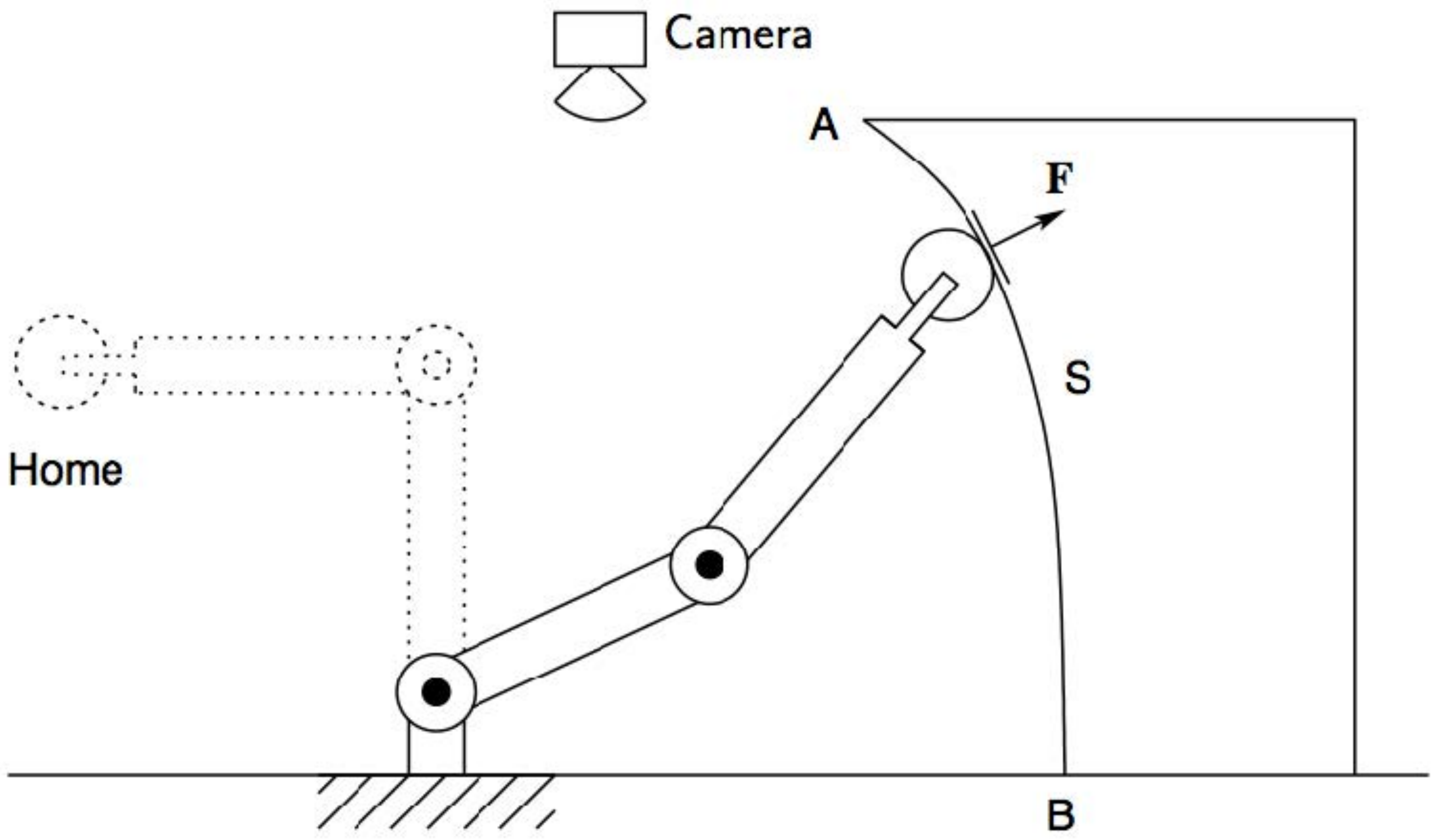
# DOF Visual Notation

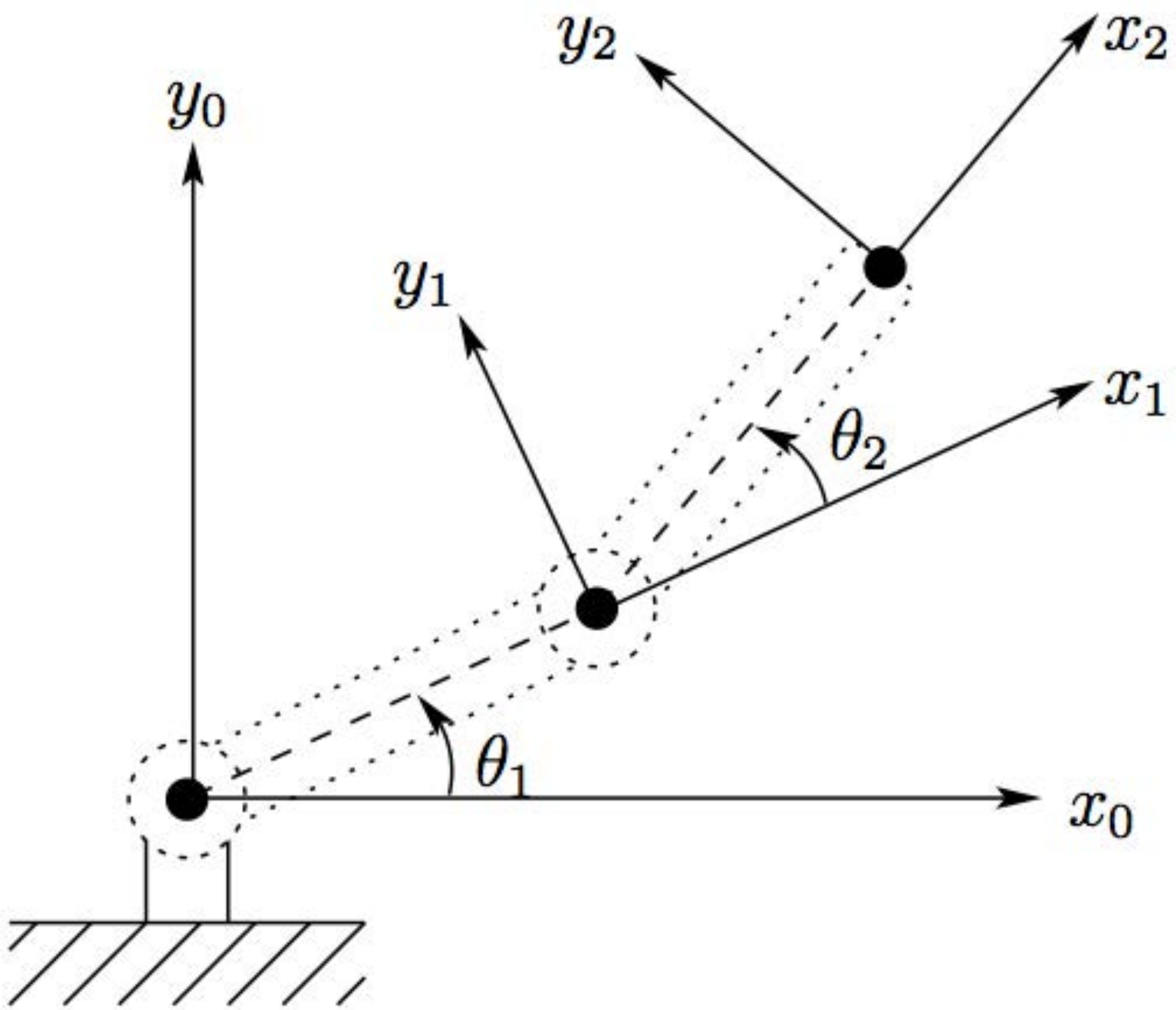





*Slide borrowed from Michigan Robotics autorob.org*

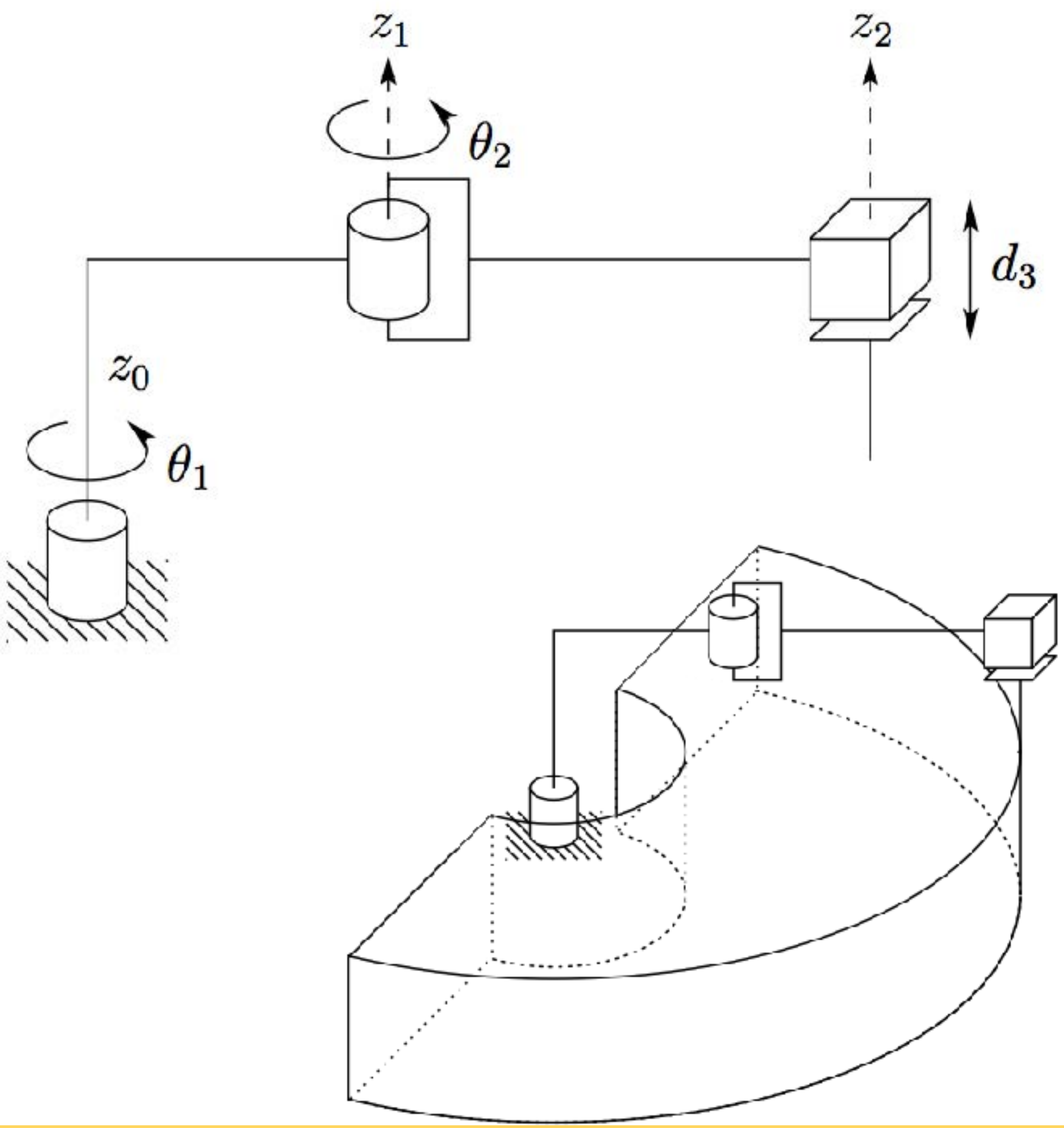
# Planar 3-link Arm (including the ground)



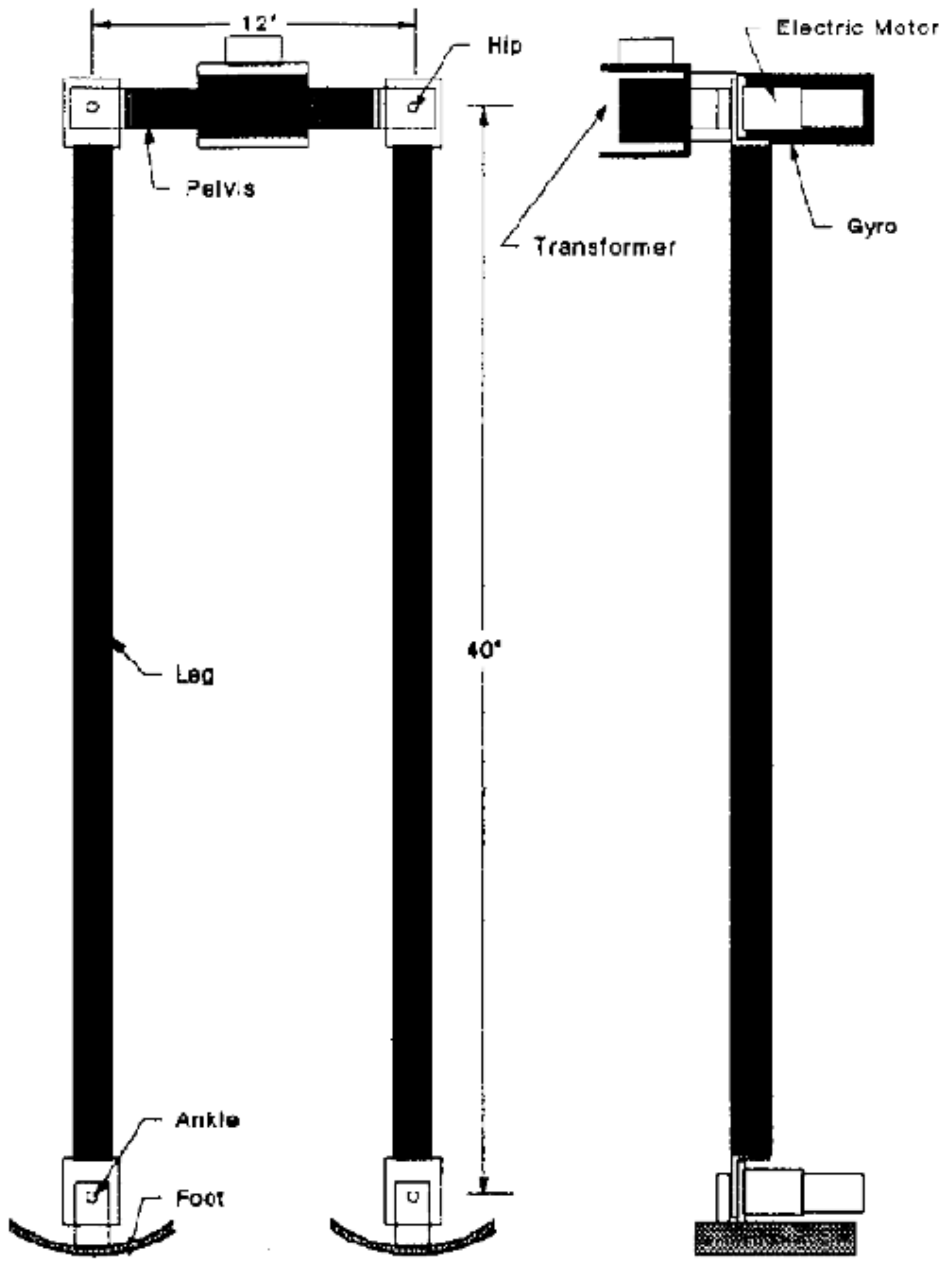
# SCARA Arm

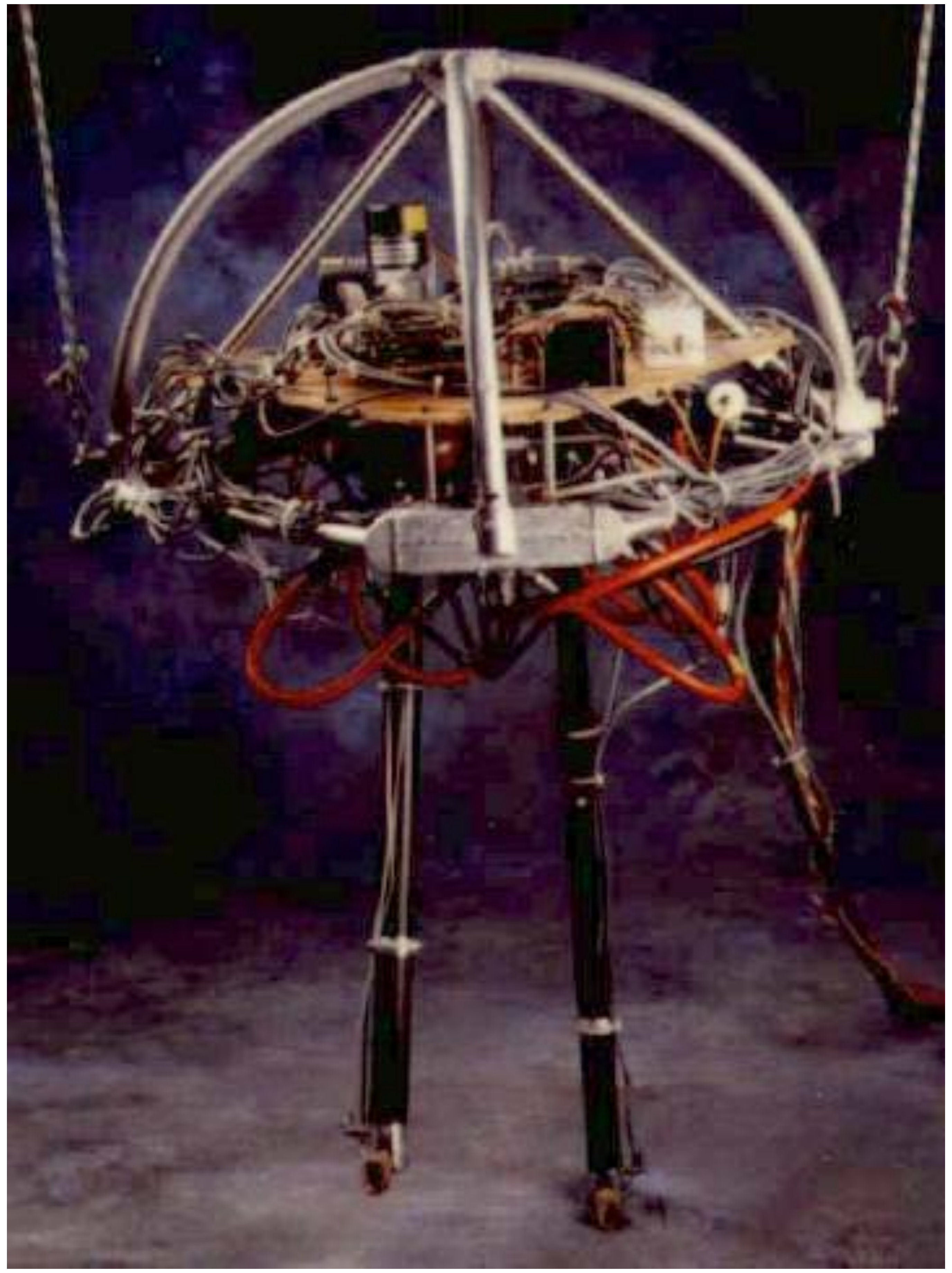
## Selective Compliance Assembly Robot Arm




https://youtu.be/7X5Nmk85kQo

# Biped Hopper (MIT Leg Lab)

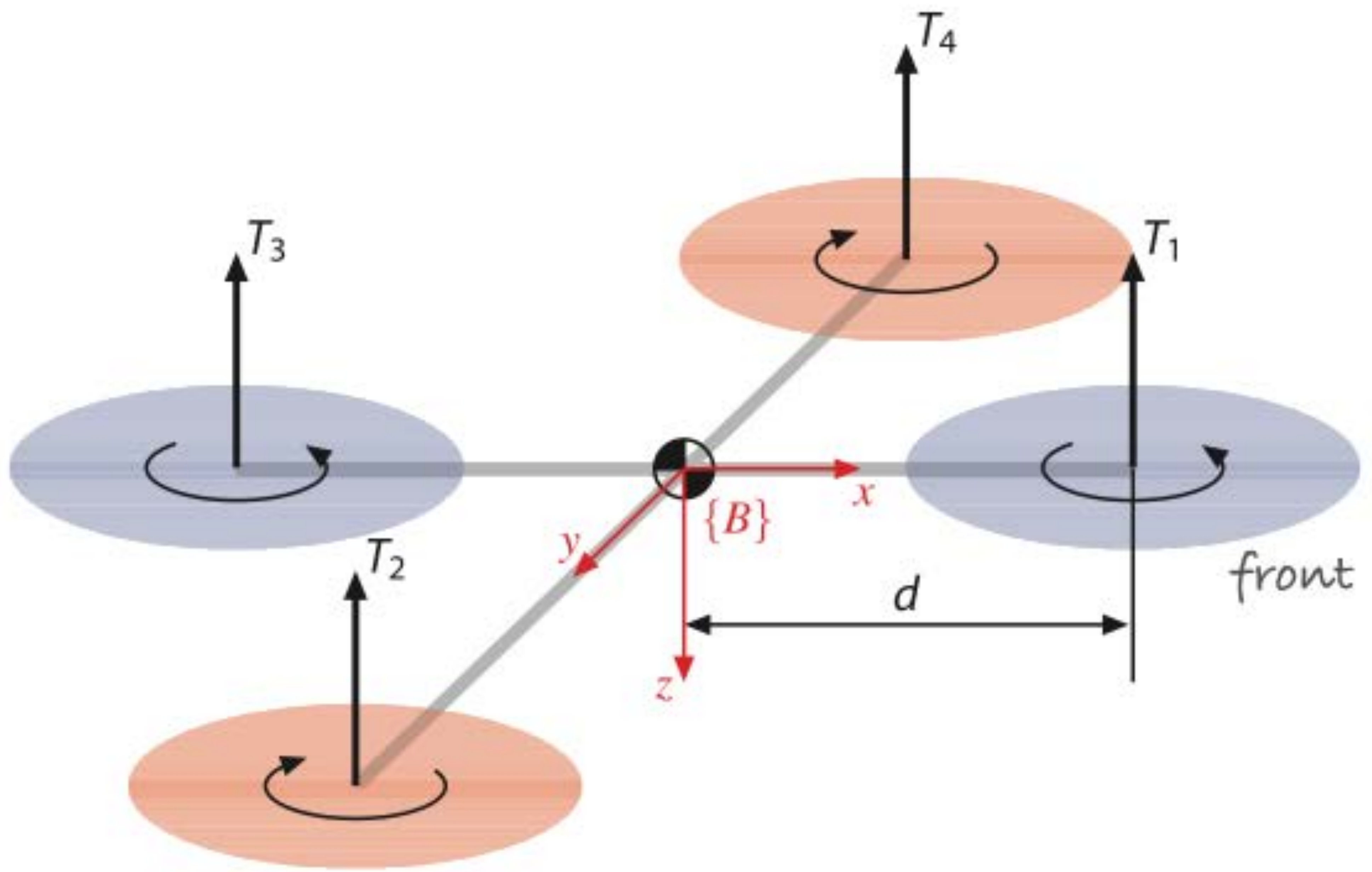


http://www.ai.mit.edu/projects/leglab/robots/robots.html

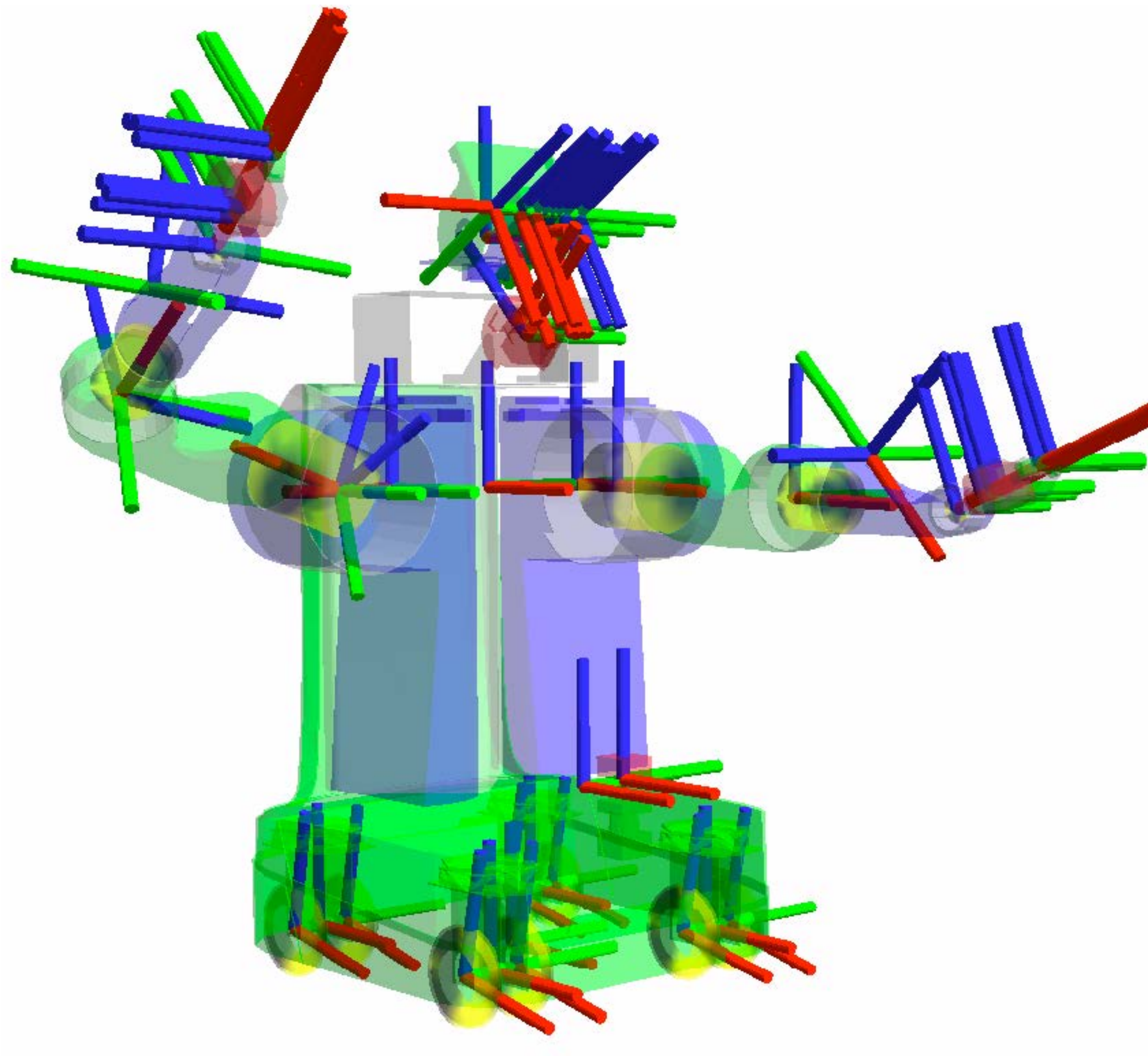
# Big Dog (BDI)

# Quad Rotor Helicopter




Safety is most important


# PR2

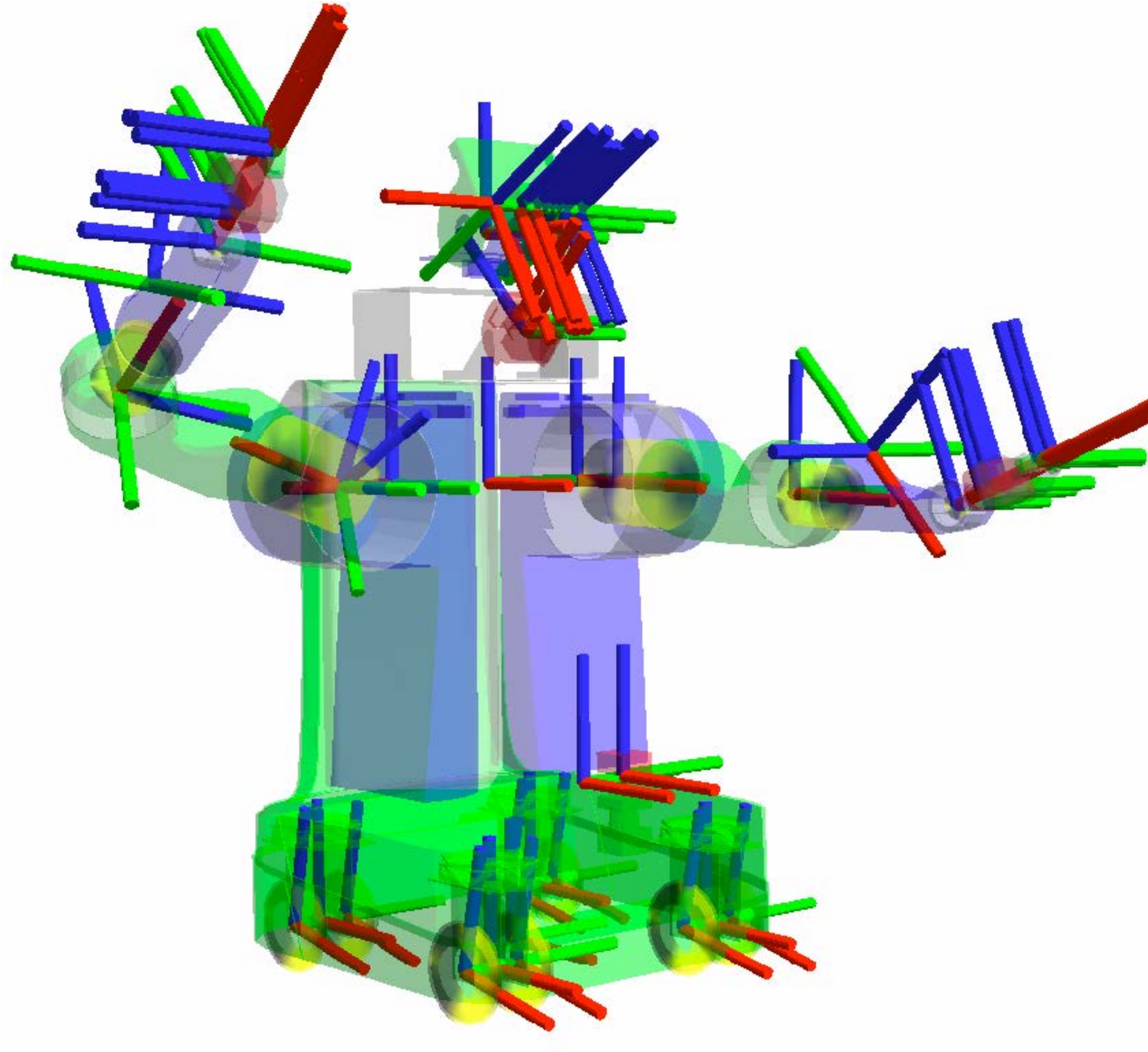


How to express kinematics as the state of an articulated system?



*Slide borrowed from Michigan Robotics autorob.org*

# PR2



How to express kinematics as the state of an articulated system?

We need some math first.

# Algebra

From Wikipedia, the free encyclopedia

**Algebra** (from Arabic *"al-jabr"* meaning "reunion of broken parts"[1]) is one of the broad parts of mathematics, together with number theory, geometry and analysis. In its most general form, algebra is the study of mathematical symbols and the rules for manipulating these symbols;[2] it is a unifying thread of almost all of mathematics.[3] As such, it includes everything from elementary equation solving to the study of abstractions such as groups, rings, and fields. The more basic parts of algebra are

What does algebra provide beyond arithmetic?

# Algebra

From Wikipedia, the free encyclopedia

- Arithmetic applies to addition and multiplication of known numbers

- Algebra includes **abstractions as variables**

  - Unknown numbers or expressions that can take on many values

- An algebra supports addition and multiplication of variables and numbers.

  - For example, from: $x^2 = 5x - 6$
  - we get: $(x - 2)(x - 3) = 0$
  - and thus: $x = 2$ or $x = 3$.

# Linear algebra

From Wikipedia, the free encyclopedia

**Linear algebra** is the branch of mathematics concerning vector spaces and linear mappings between such spaces. Such an investigation is initially motivated by a system of linear equations containing several unknowns. Such equations are naturally represented using the formalism of matrices and vectors.[1]

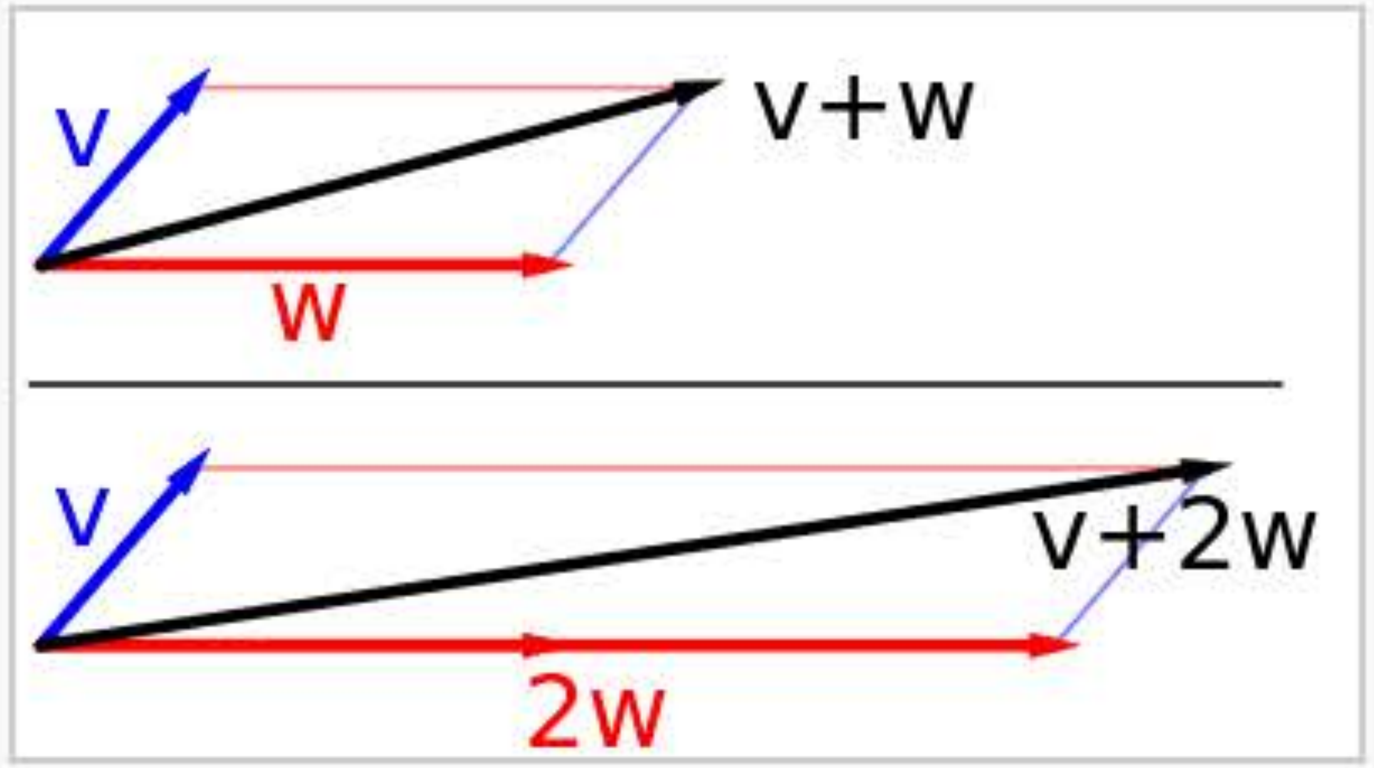
What does is linear algebra provide beyond algebra?

# Vector space

From Wikipedia, the free encyclopedia

*This article is about linear (vector) spaces. For the structure in incidence geometry, see Linear space (geometry).*

A **vector space** (also called a **linear space**) is a collection of objects called **vectors**, which may be added together and multiplied ("scaled") by numbers, called *scalars* in this context. Scalars are often taken to be real numbers, but there are also vector spaces with scalar multiplication by complex numbers, rational numbers, or generally any field. The operations of vector addition and scalar multiplication must satisfy certain requirements, called *axioms*, listed below.




Vector addition and scalar multiplication: a vector **v** (blue) is added to another vector **w** (red, upper illustration). Below, **w** is stretched by a factor of 2, yielding the sum $\mathbf{v} + 2\mathbf{w}$.

- Describes spaces where vector operations are closed with respect to:
  - [redacted]
  - [redacted]

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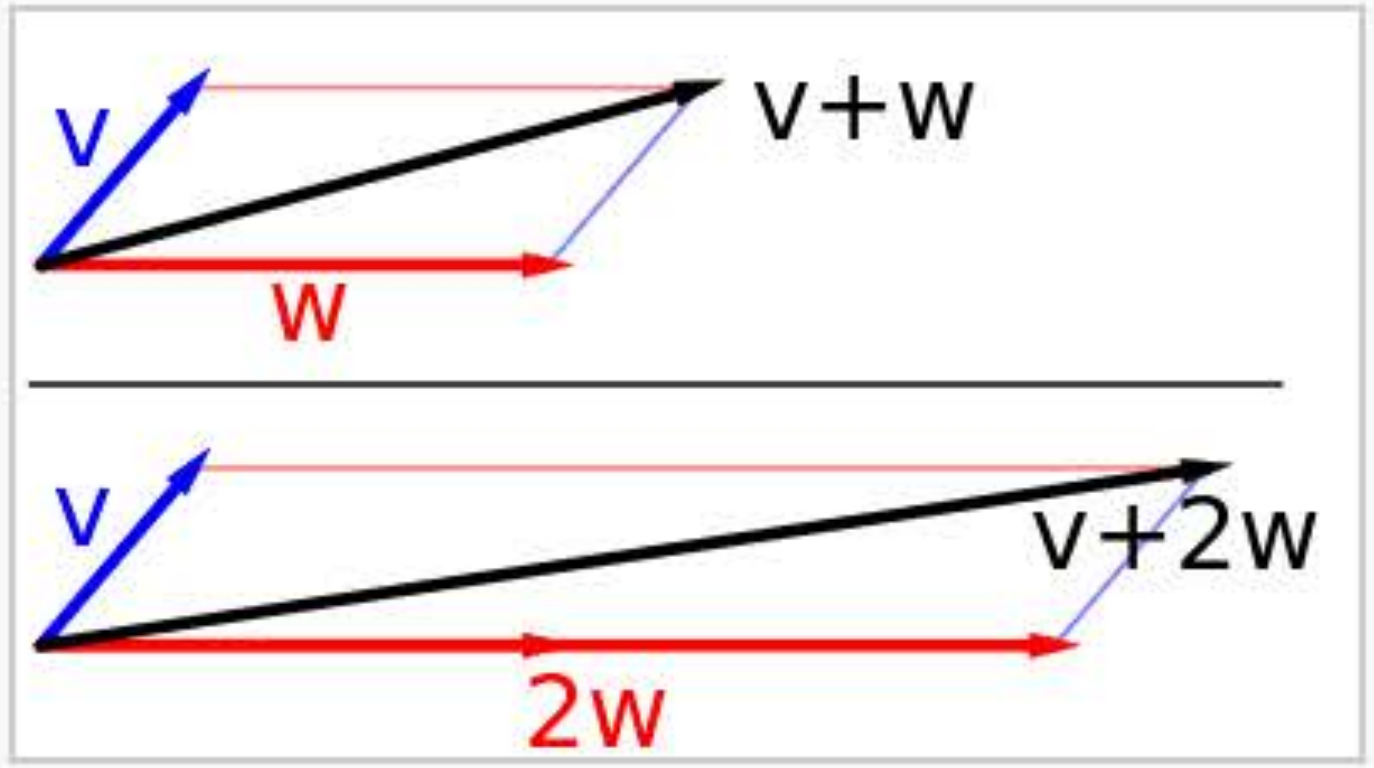


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- Describes spaces where vector operations are closed with respect to:
  - addition
  - scalar multiplication

| | Arithmetic | Algebra | Linear Algebra |
|---|---|---|---|
| **Abstraction** | | $x = 3$ | $x = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$ |
| **Addition** | $3 + 2 = 5$ | $x + 2 = 5$ | $x + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}$ |
| **Scalar multiplication** | $3 \times 2 = 6$ | $2x = 6$ | $2x = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$ |

# Linear algebra

From Wikipedia, the free encyclopedia

- Many important complex systems are described by collections of linear equations.

- An algebra of scalars, vectors, and matrices helps us work with these systems, keeping track of the complexity.

  - Manipulate groups of known and unknown parameters, just like manipulating numbers.

- Linear algebra is essential for representing frames of reference, rotation, translation, and general 3D homogeneous transforms.

# Linear Algebra (Rough) Breakdown

- Geometry of Linear Algebra ← *primary focus for 5551*
  - Vectors, matrices, basic operations, lines, planes, homogeneous coordinates, transformations
- Solving Linear Systems ← *needed for iterative IK*
  - Gaussian Elimination, LU and Cholesky decomposition, over-determined systems, calculus and linear algebra, non-linear least squares, regression
- The Spectral Story
  - Eigensystems, singular value decomposition, principle component analysis, spectral clustering

# Systems of Linear Equations

# Linear algebra

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$$\begin{aligned} 3x + 2y - \phantom{4}z &= 1 \\ 2x - 2y + 4z &= -2 \\ -x + \tfrac{1}{2}y - \phantom{4}z &= 0 \end{aligned}$$

is solved by [redacted]

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linear systems expressed in general matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

as

$$\begin{bmatrix} 3 & 2 & -1 \\ 2 & -2 & 4 \\ -1 & \frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

M linear equations

each equation yields a hyperplane in N-D

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$$A \quad \mathbf{x} = \mathbf{b}$$

vector of N unknowns to be found

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If #unknowns > #equations, [redacted]

If #unknowns < #equations, [redacted]

If #unknowns = #equations, [redacted]

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If #unknowns > #equations, underdetermined system, usually with infinite solutions

If #unknowns < #equations, [redacted]

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vector of N unknowns to be found

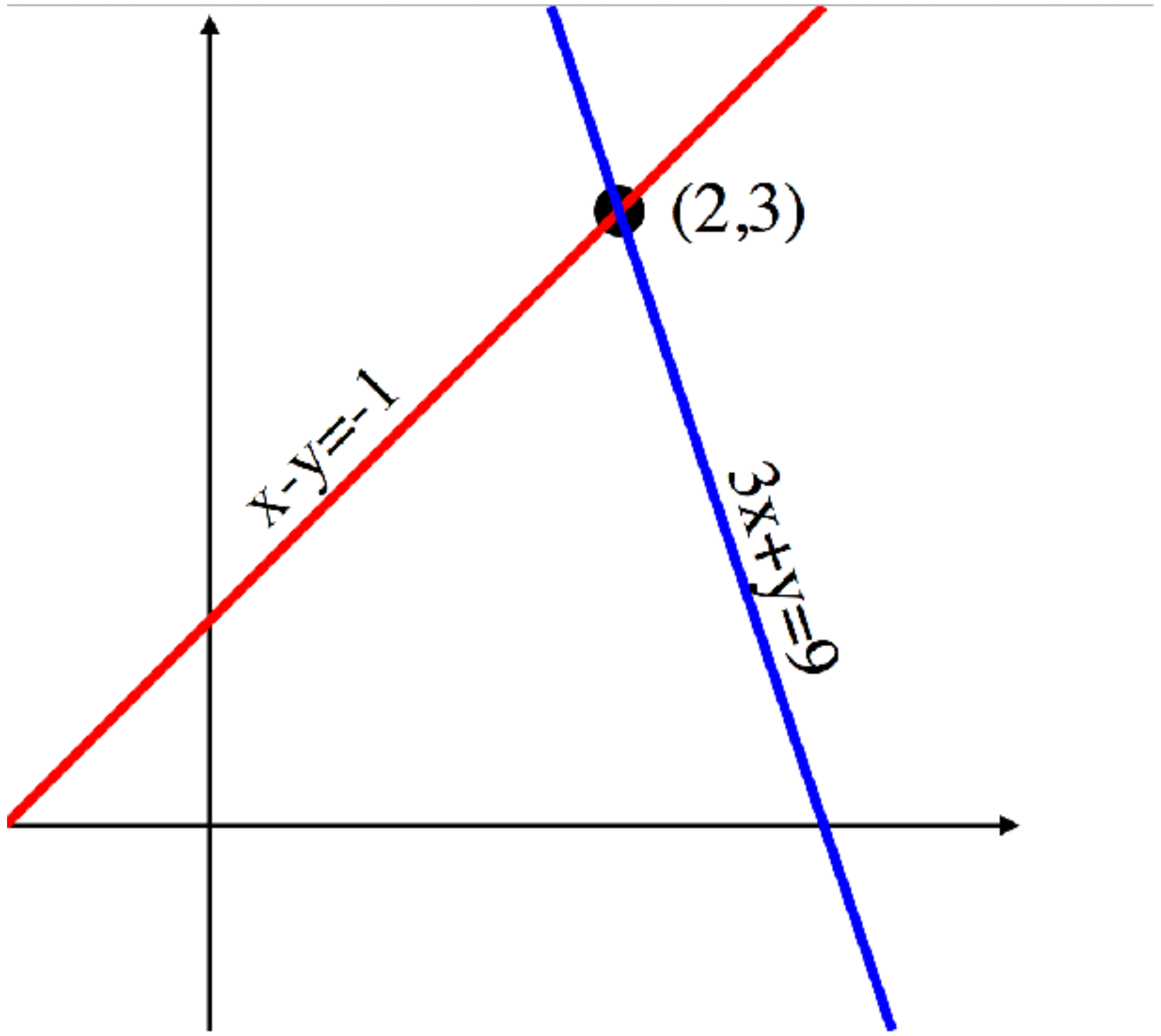
If #unknowns > #equations, underdetermined system, usually with infinite solutions

If #unknowns < #equations, overdetermined system, usually with no solutions

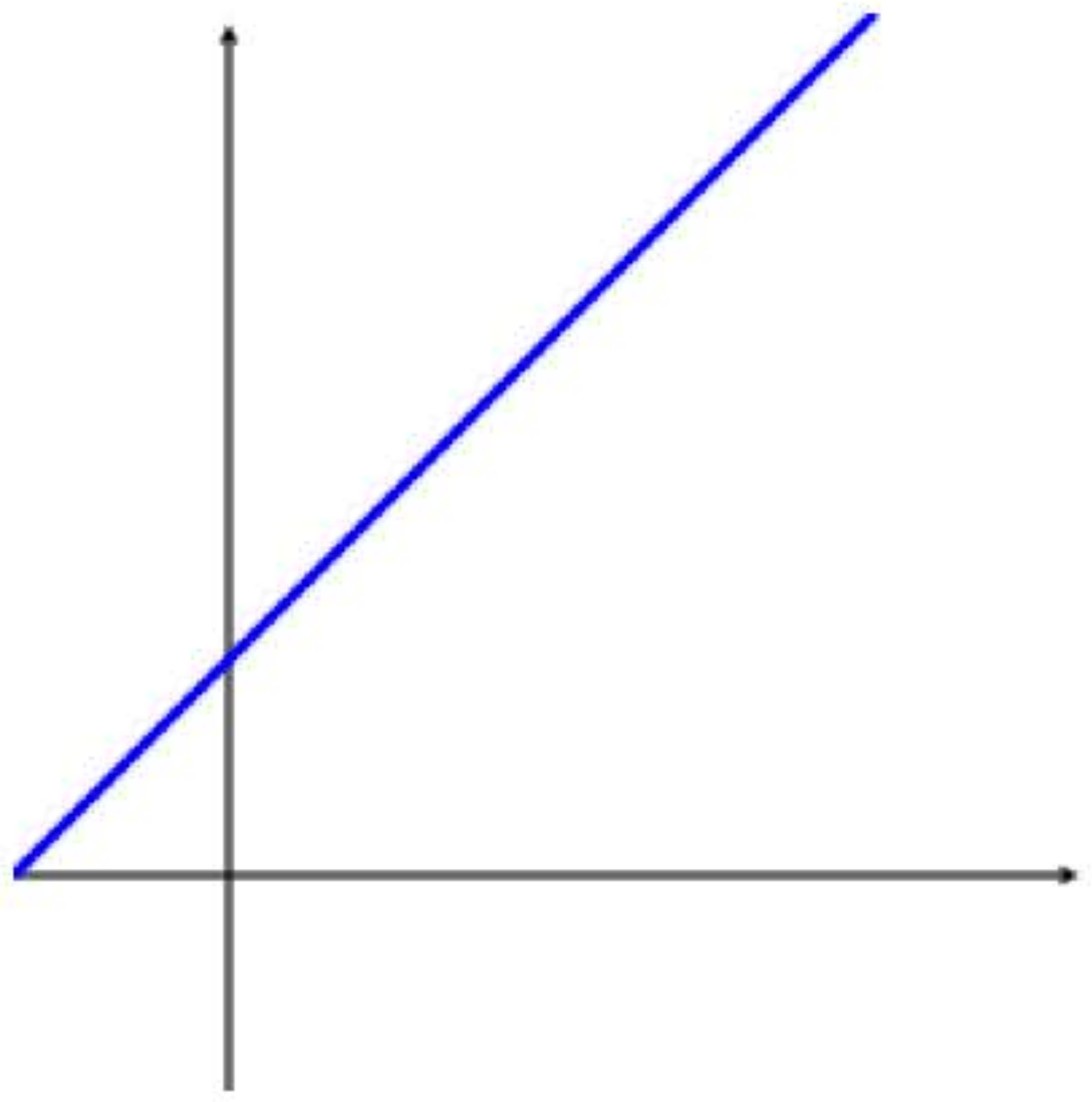
If #unknowns = #equations, usually has a unique solution

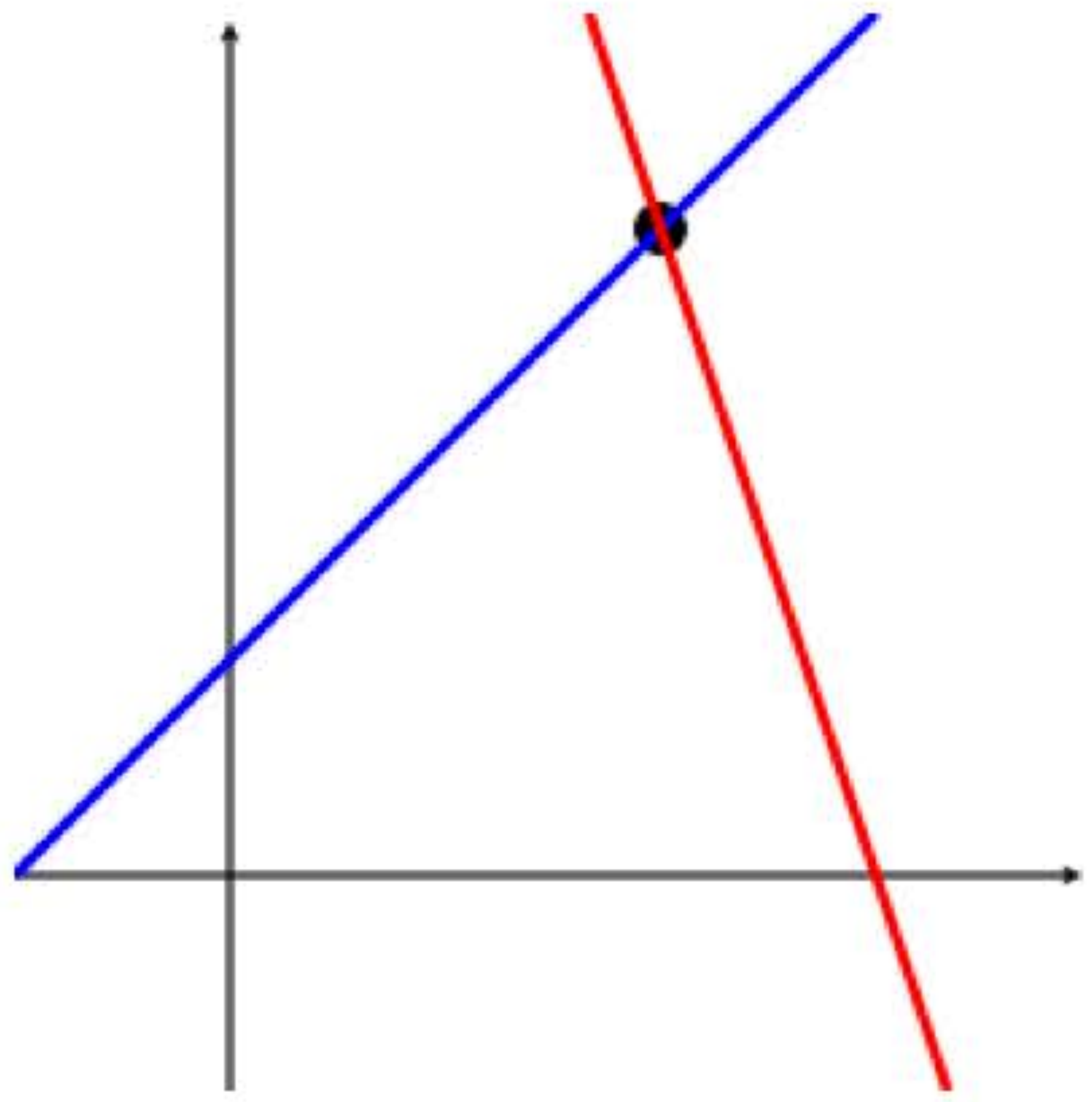
# 2D Example

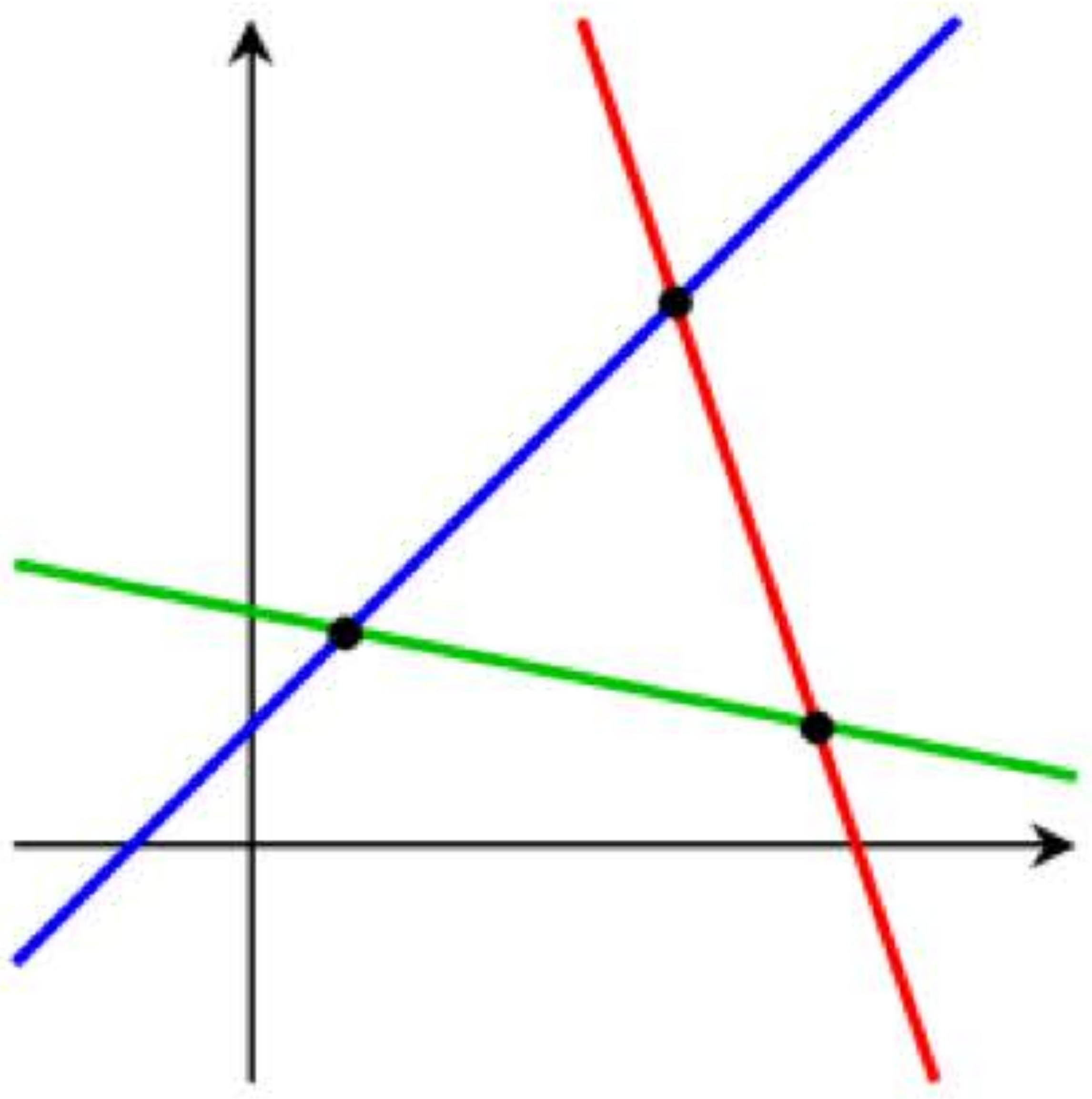
only single point
satisfies both lines



# 2D Example


any point on the line satisfies
One equation

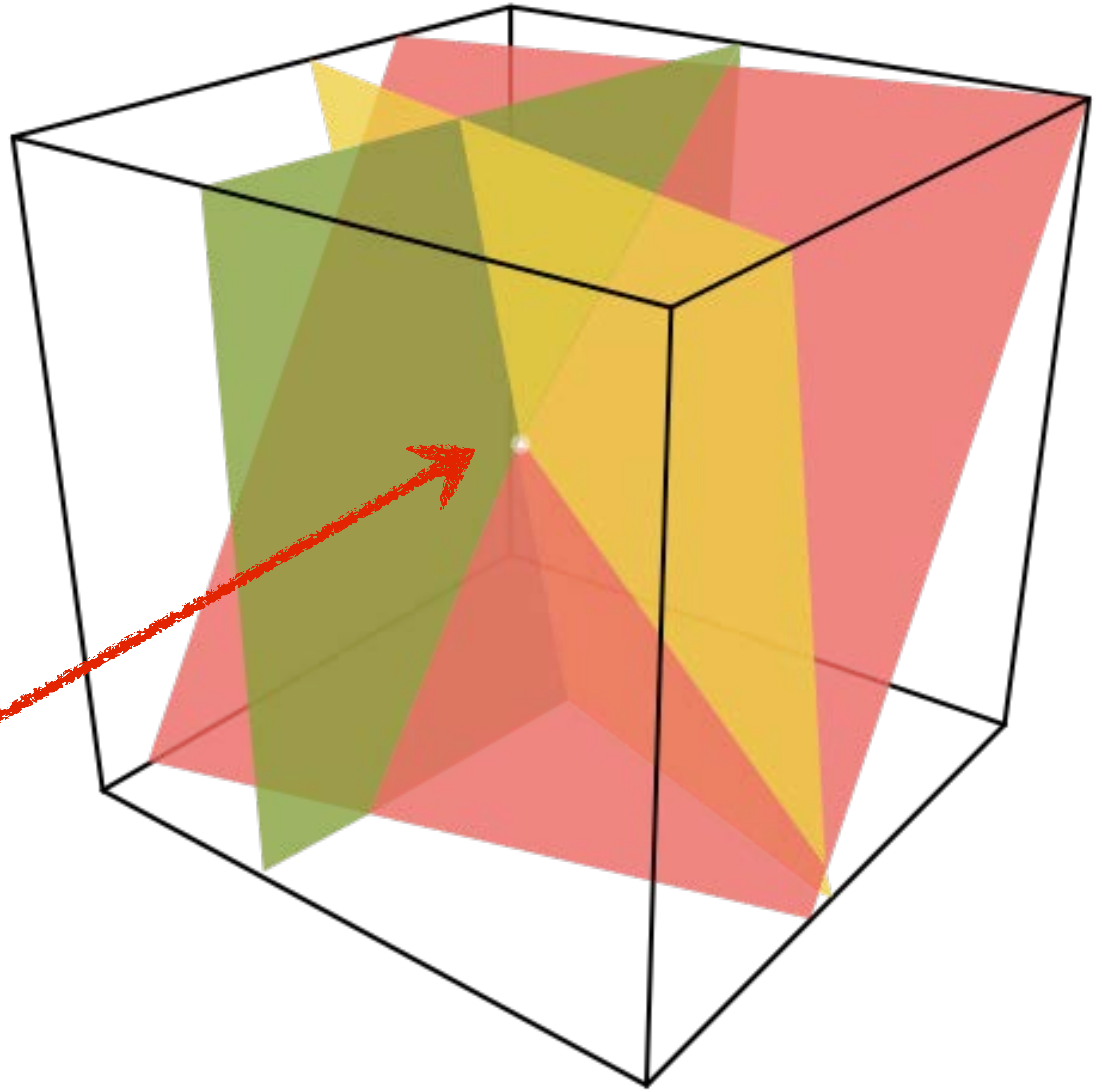

only single point satisfies both lines
Two equations


no point satisfies all three lines
Three equations

# 3D Example

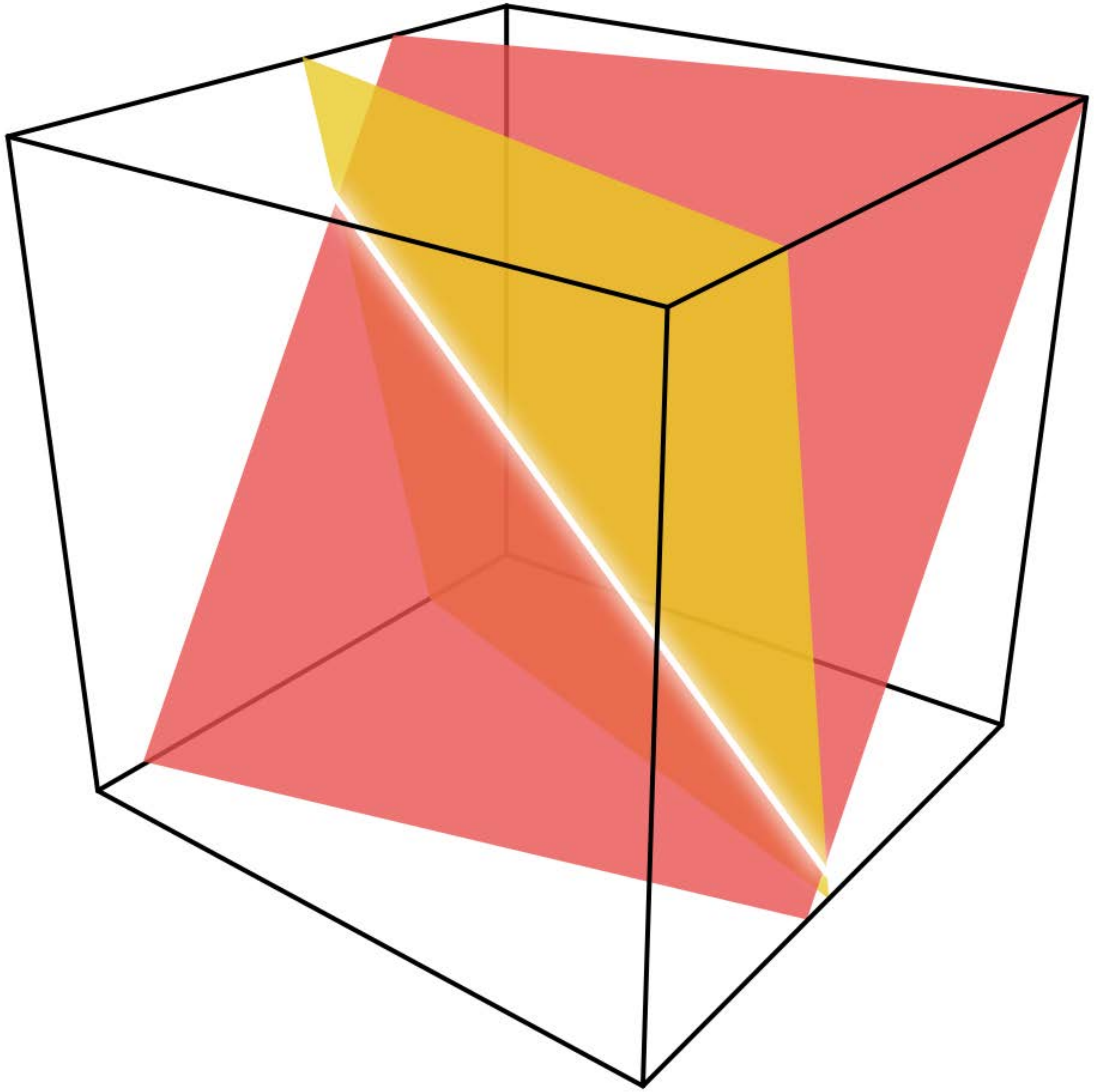
Each equation yields a 2D plane in 3D space

A single point satisfies all equations



# 3D Example

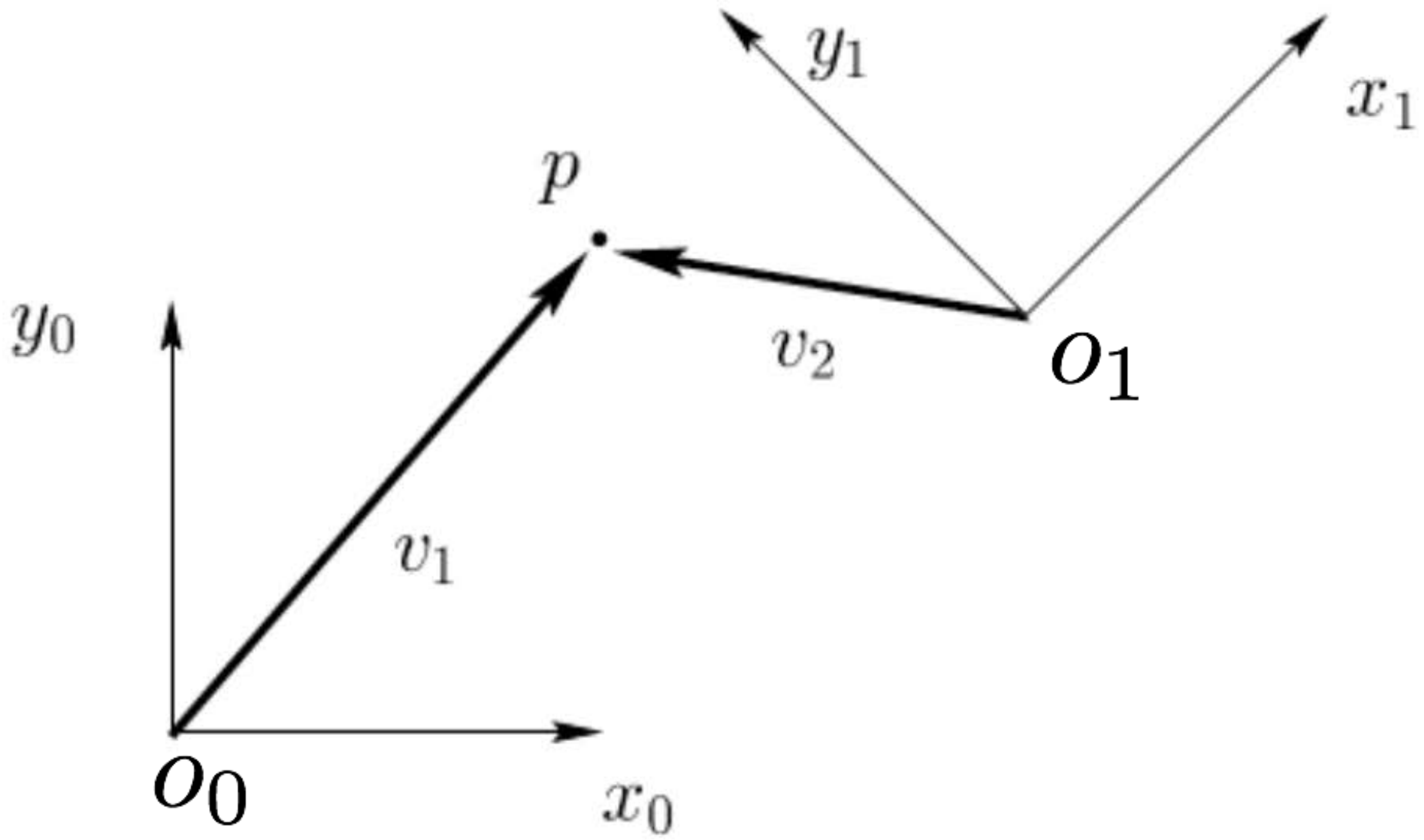
How many solutions?



# Coordinate Spaces as Systems of Linear Equations

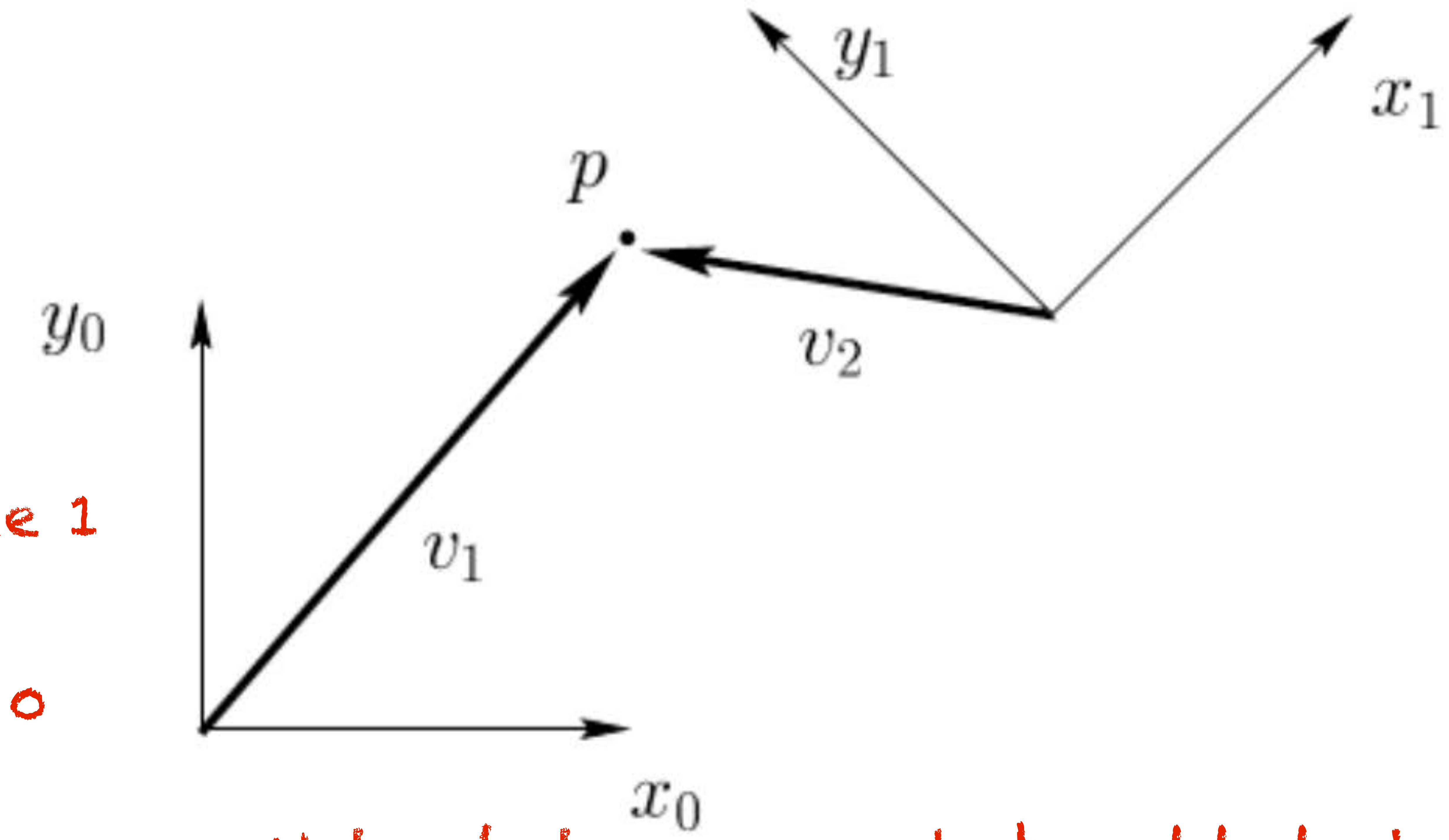
# Coordinate Spaces (2D)

- Two coordinate frames $o_0x_0y_0$ and $o_1x_1y_1$, and a point $p$.
- The location of point p can be described with respect to either coordinate frame: $p^0 = [5, 6]^T$ and $p^1 = [-2.8, 4.2]^T$.
- The vector $v_1$ is direction and magnitude from $o_0$ to $p$, and $v_2$ is from $o_1$ to $p$.



# Coordinate Spaces (2D)

- Point *p* has a location.
- Vectors $v_1$ and $v_2$ have directions and magnitudes.
- $v_1^0 = [5, 6]^T$ vector 1 in frame 0
- $v_1^1 = [7.77, 0.8]^T$ vector 1 in frame 1
- $v_2^0 = [-5.1, 1]^T$ vector 2 in frame 0
- $v_2^1 = [-2.8, 4.2]^T$ vector 2 in frame 1




Note: Vectors can only be added when they are in the same coordinate frame.

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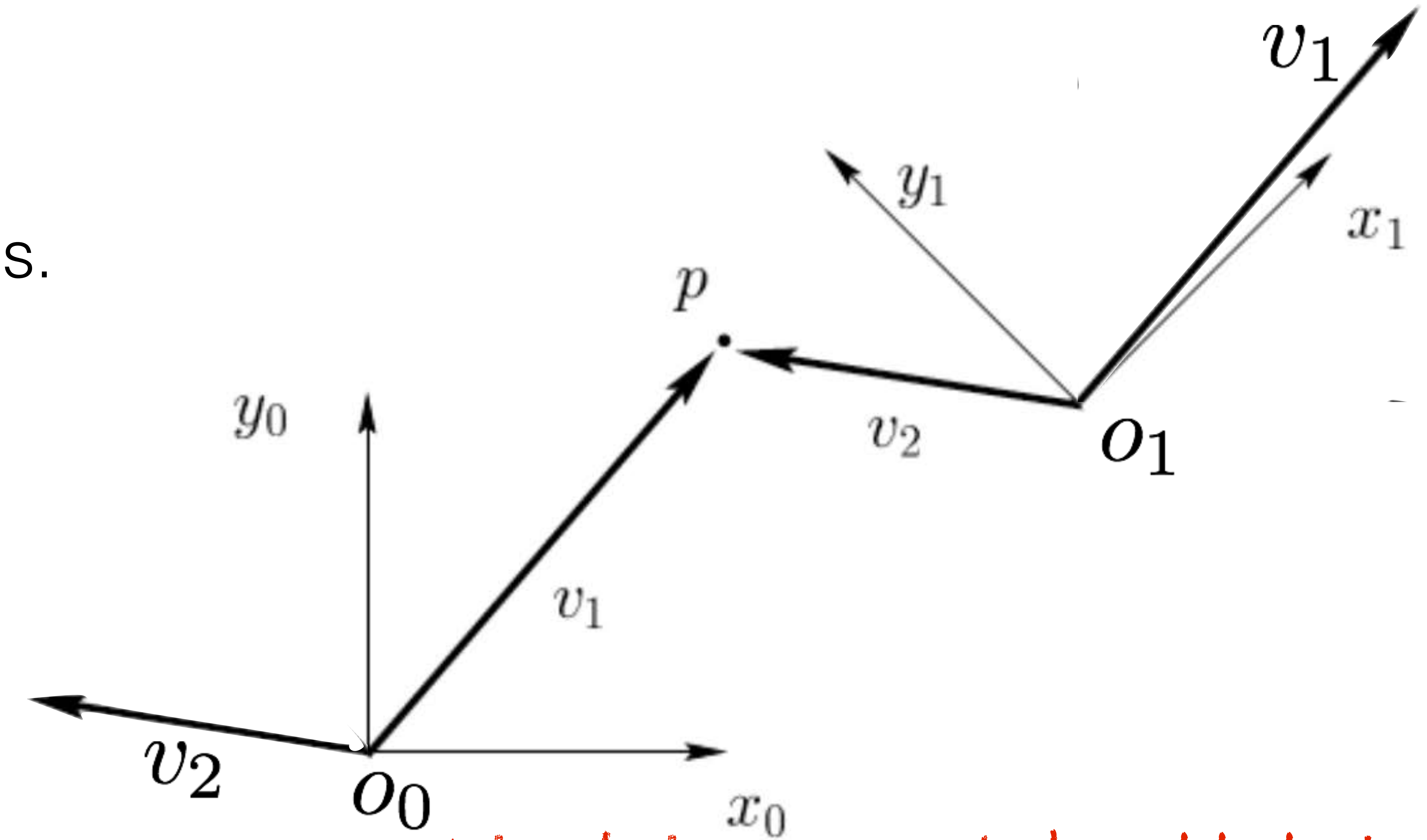


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# Vectors and Matrices

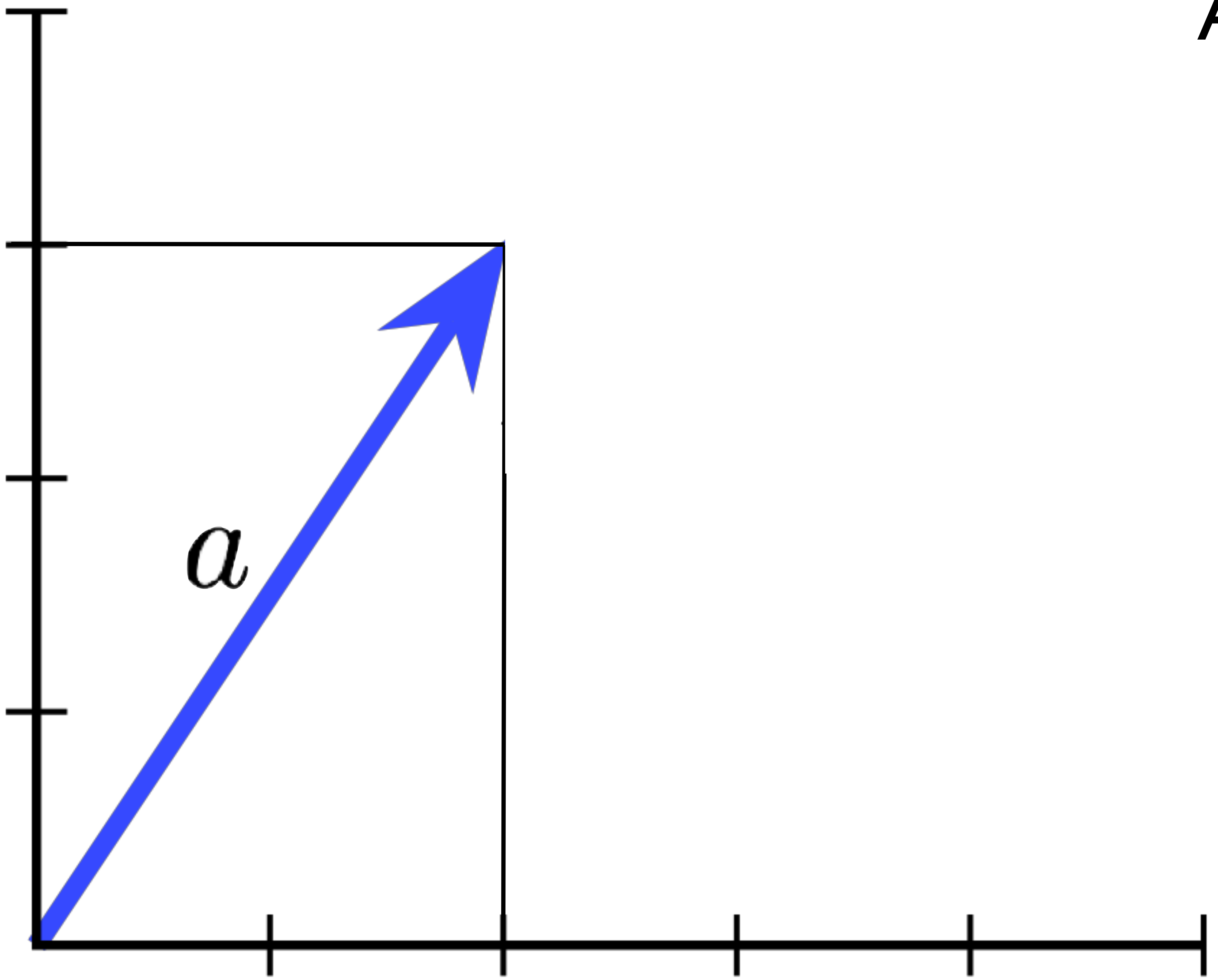
N-dimensional vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

M-by-N matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

# 2D Vector

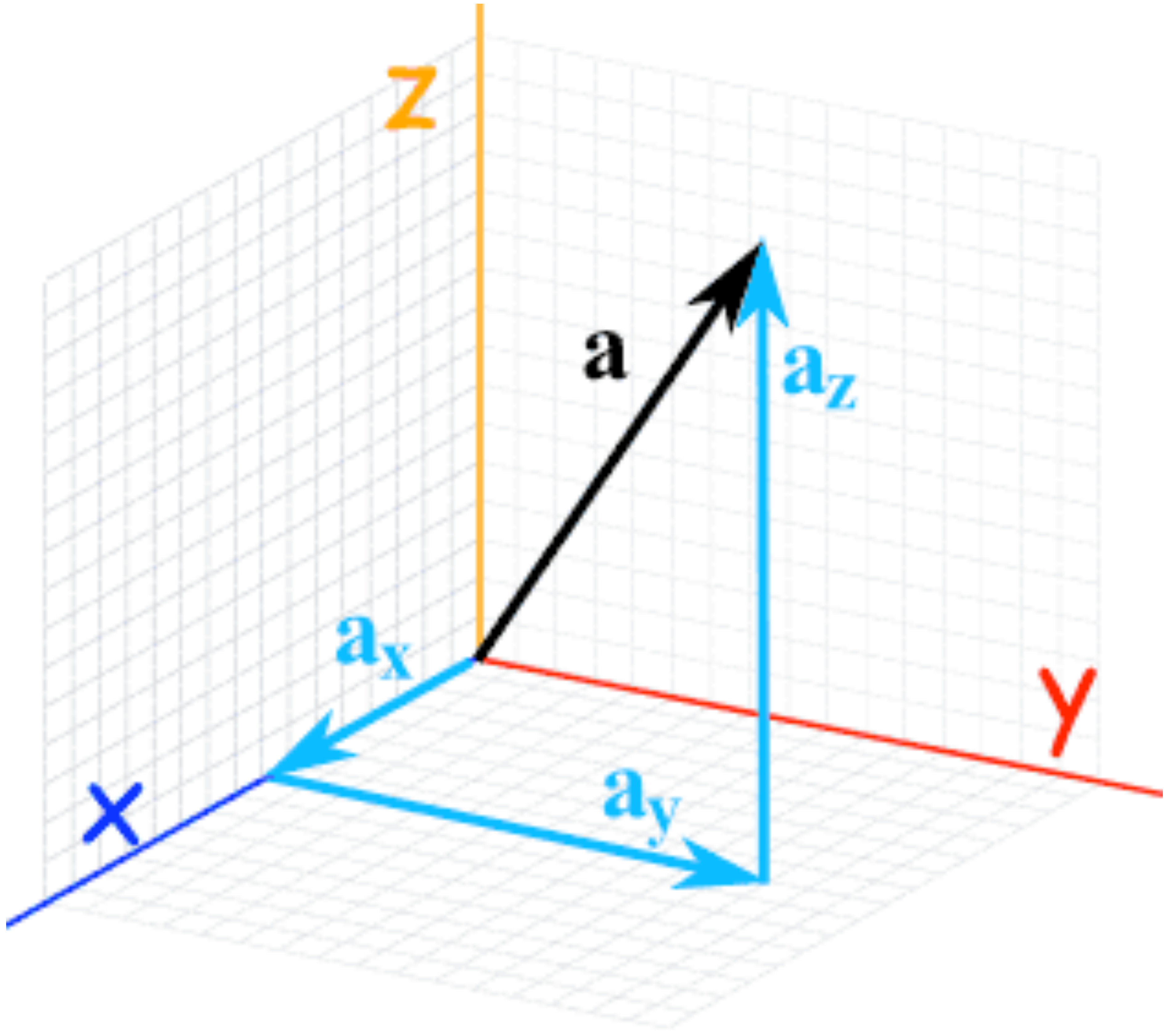



A vector is a motion in space

$$a = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

```
var a = [ [2],
          [3] ];
```

# 3D Vector




$$a = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

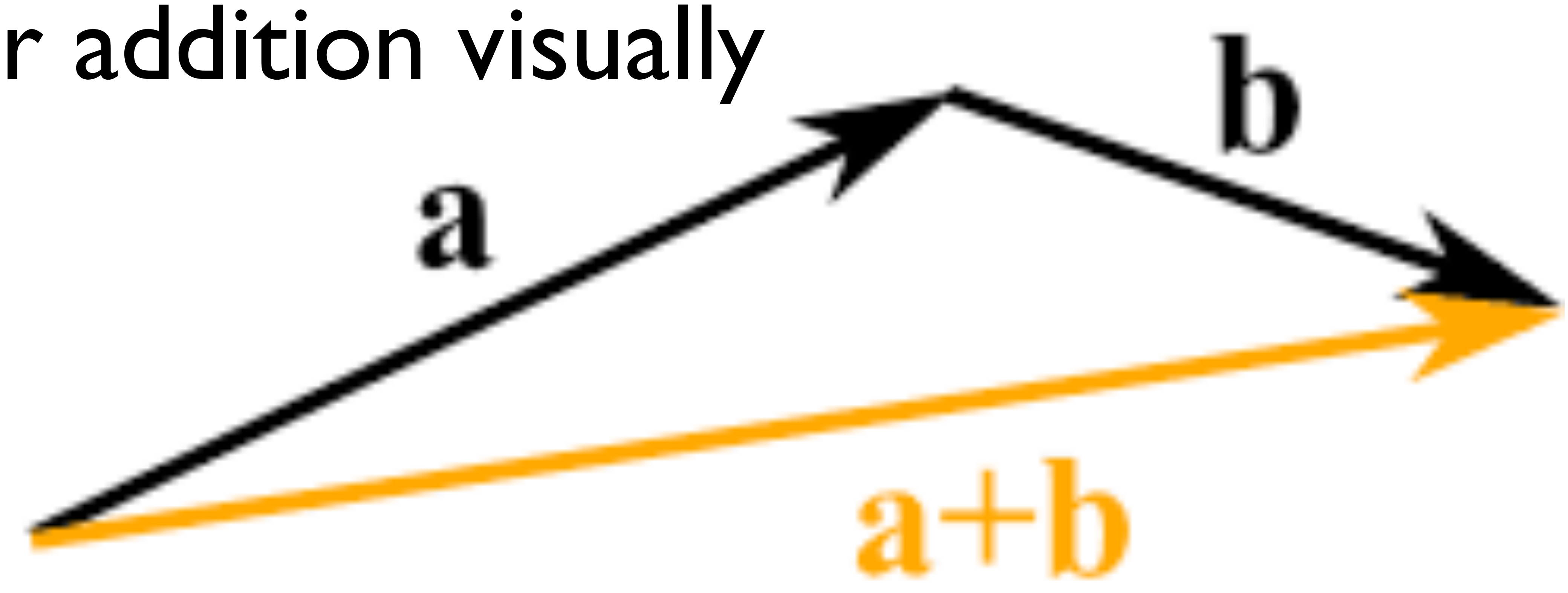
```
var a = [ [ax],
          [ay],
          [az] ];
```

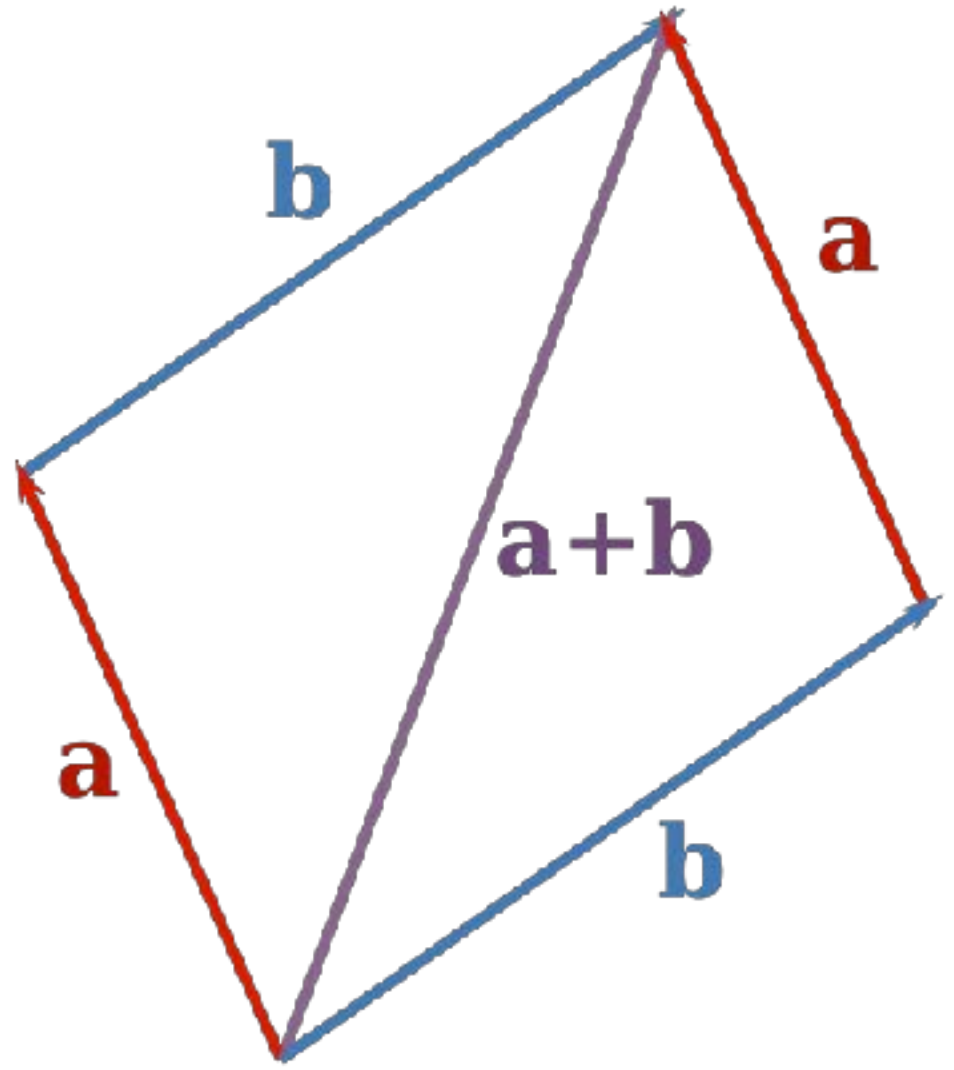
# Vector Addition and Subtraction

$$a + b = \begin{bmatrix} a_x + b_x \\ a_y + b_y \\ a_z + b_z \end{bmatrix}$$

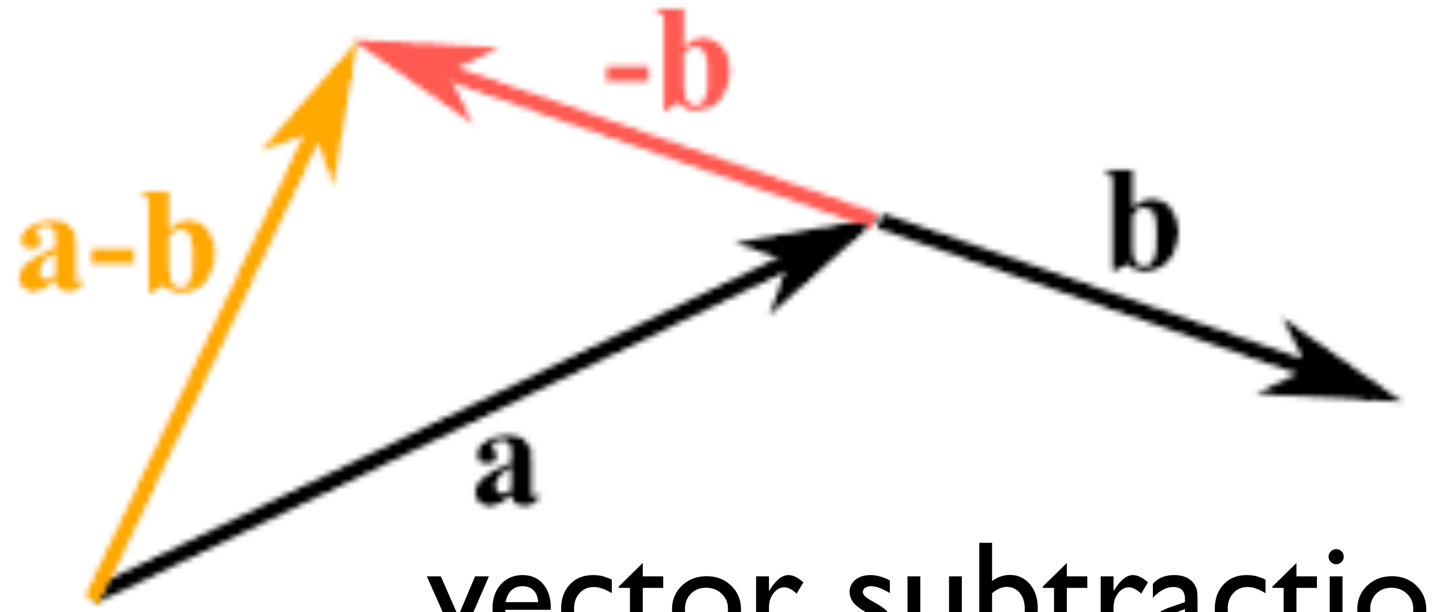
vector result

vector addition visually




vector addition is order independent




vector subtraction is addition with negated vector

# Magnitude and Unit Vector

The magnitude of a vector is the square root of the sum of squares of its components

$$||a|| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

A unit vector has a magnitude of one.
Normalization scales a vector to unit length.

$$\hat{a} = \frac{a}{||a||}$$

A vector can be multiplied by a scalar

$$sa = \begin{bmatrix} sa_x \\ sa_y \\ sa_z \end{bmatrix}$$

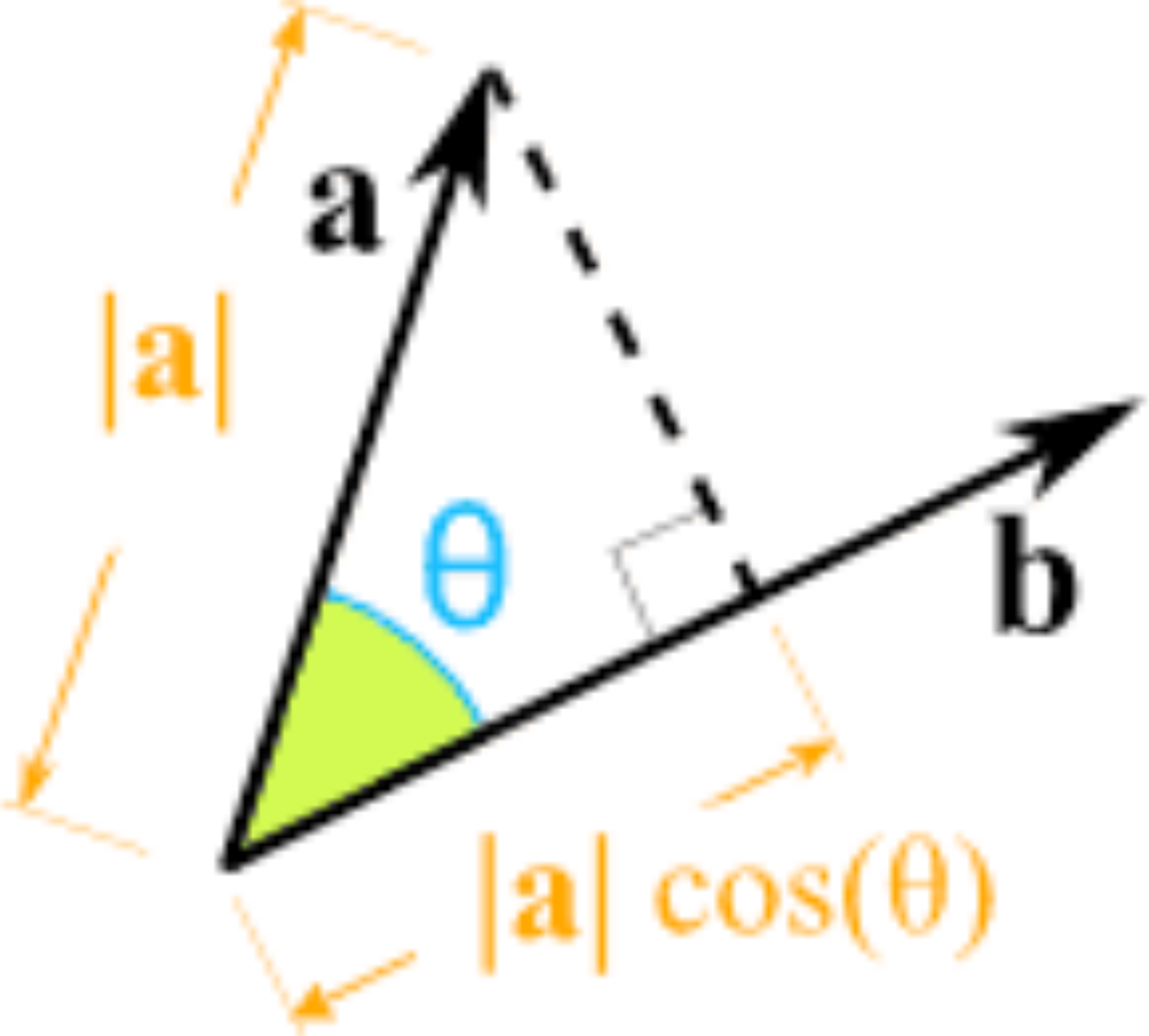
scalar result

# Dot Product

$$a \bullet b = a_x b_x + a_y b_y + a_z b_z$$
$$= ||a|| ||b|| cos(\theta)$$

Measures the similarity in direction of two vectors

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 2 * 3 + 1 * 2 = 8$$

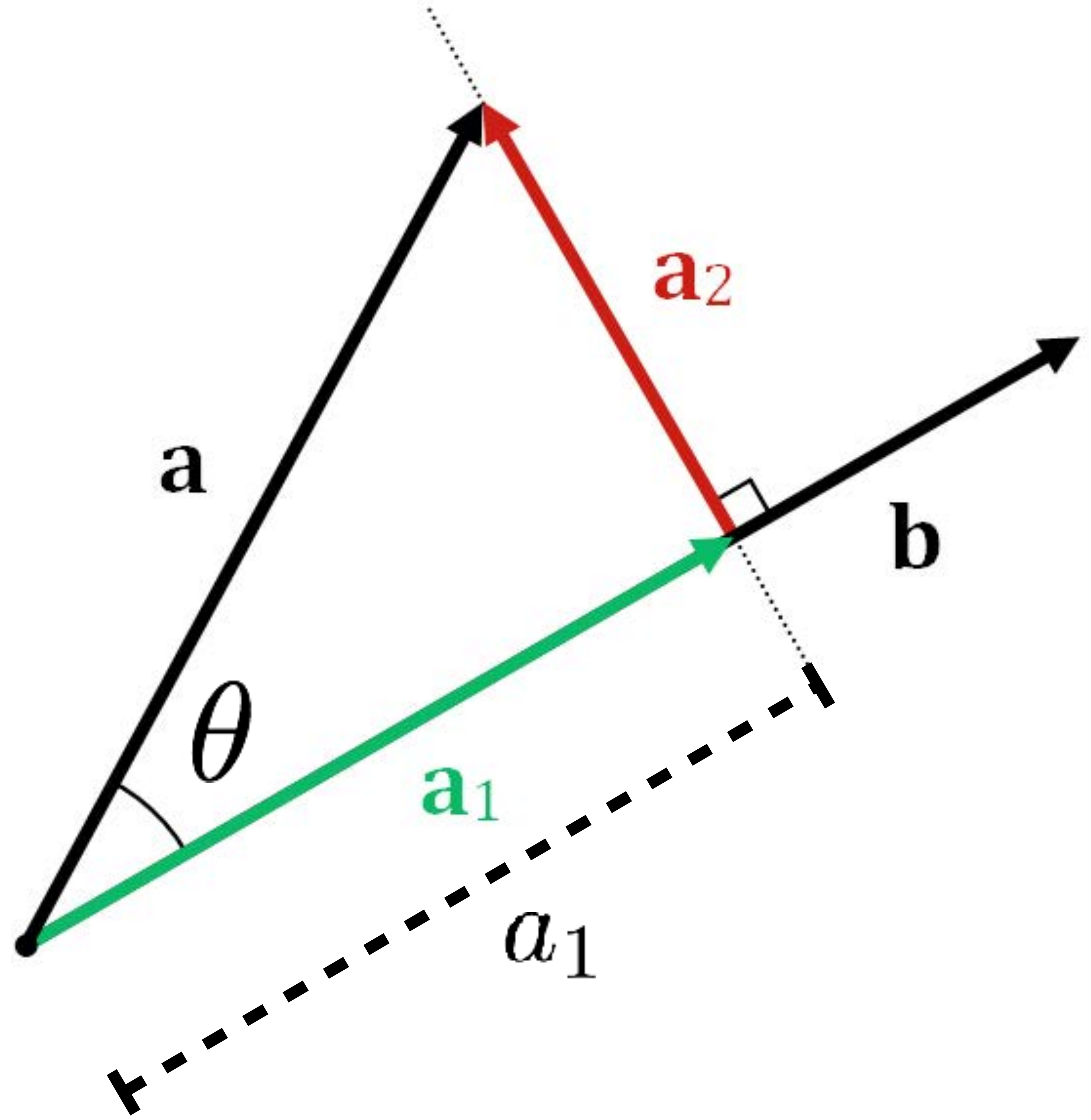


# Projections

Dot products related to projections onto vectors.

Scalar projection of one vector onto another

$$a_1 = |\mathbf{a}| \cos\theta = \mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \frac{\mathbf{b}}{|\mathbf{b}|}$$




Vector projection

$$\mathbf{a}_1 = a_1 \hat{\mathbf{b}}$$

$\hat{\mathbf{b}}$ is unit vector



Slide borrowed from Michigan Robotics autorob.org

# Checkpoint

- What is the dot product of a vector with itself?

[redacted]

- What is the dot product of two orthogonal vectors?

[redacted]

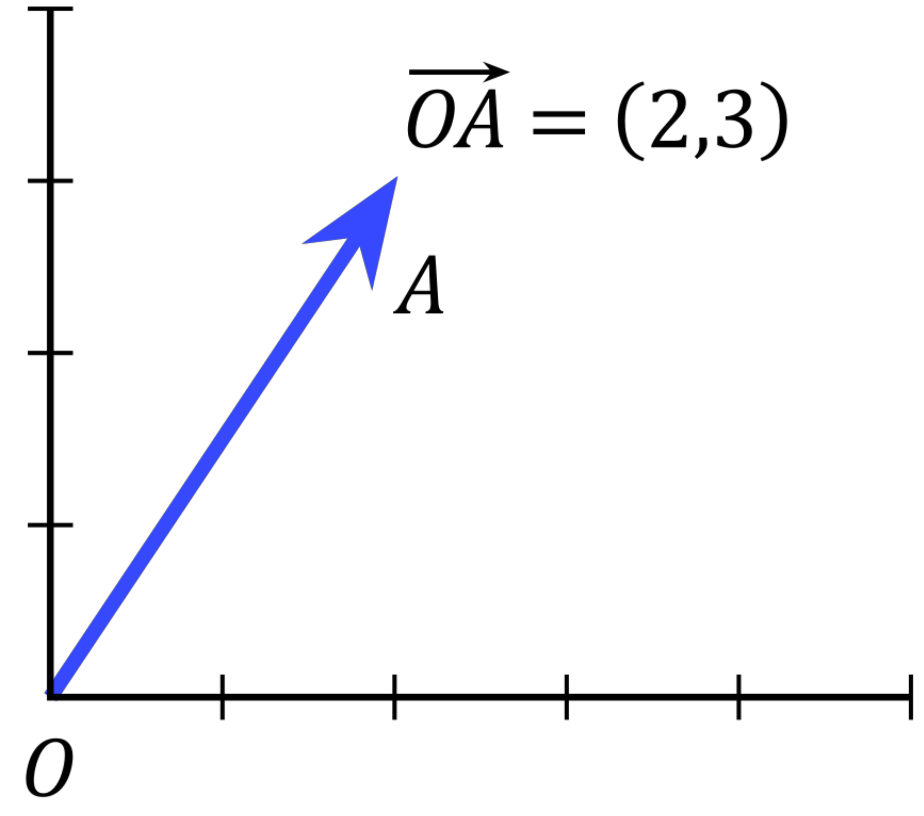
# Checkpoint

- What is the dot product of a vector with itself?
  - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?
  - [redacted]

# Checkpoint

- What is the dot product of a vector with itself?
  - the square of the vector magnitude
- What is the dot product of two orthogonal vectors?
  - 0

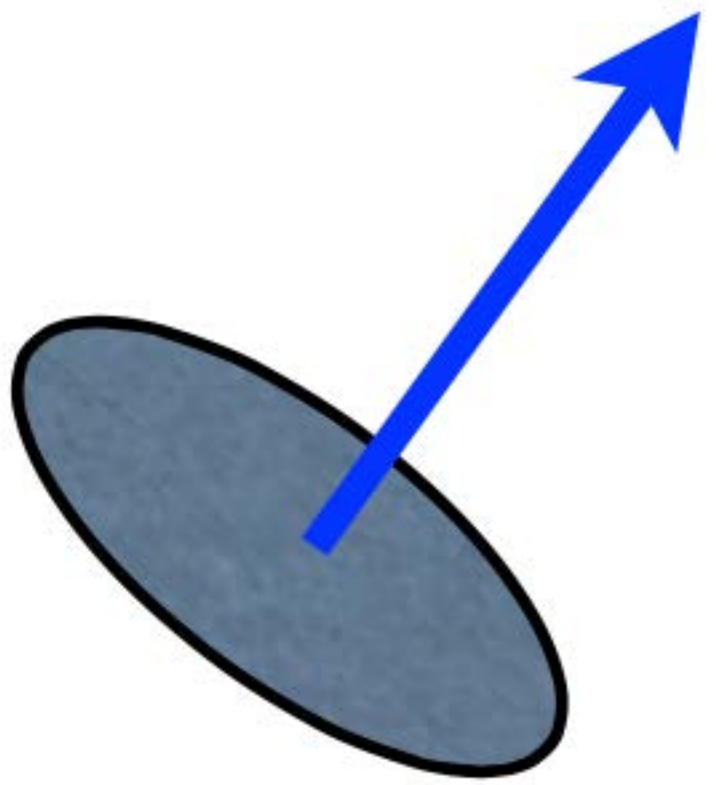
# Checkpoint




- How many unit vectors are perpendicular to a 2D vector?

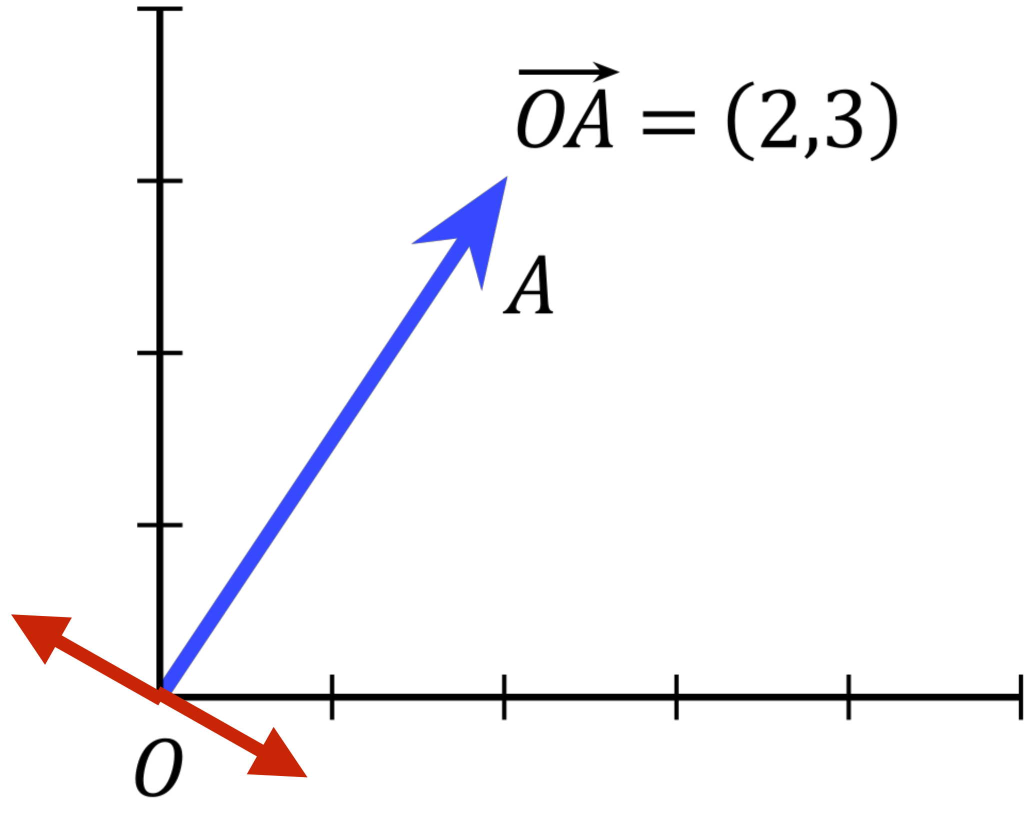
[redacted]

- How many unit vectors are perpendicular to a 3D vector?



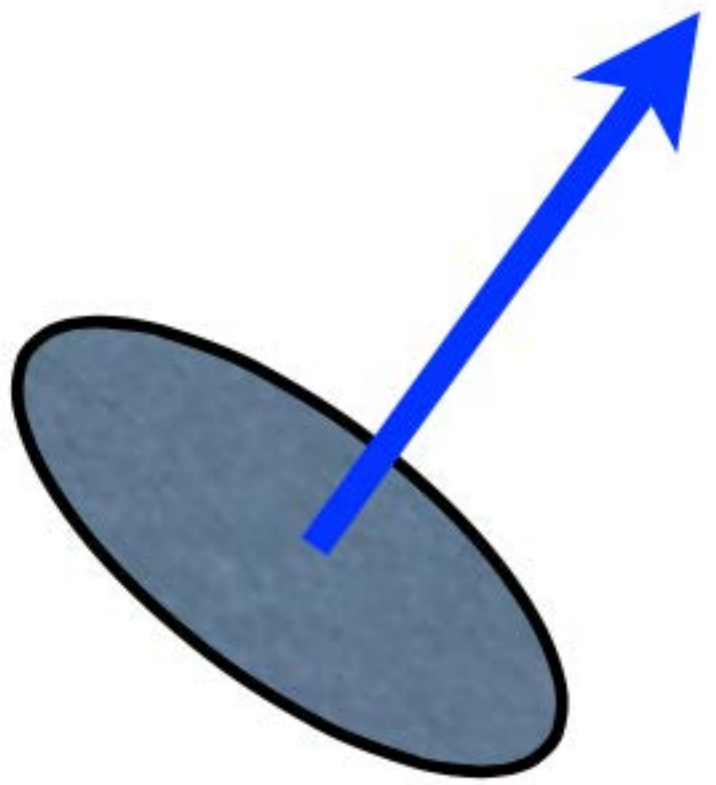
[redacted]

# Checkpoint

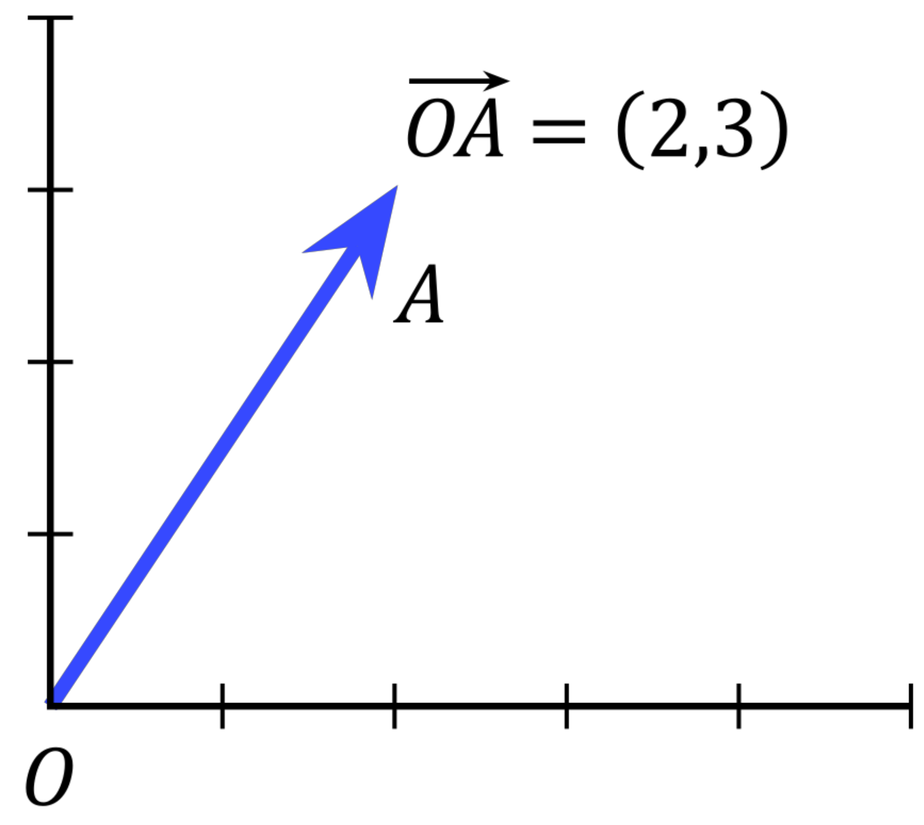



- How many unit vectors are perpendicular to a 2D vector?
    - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?

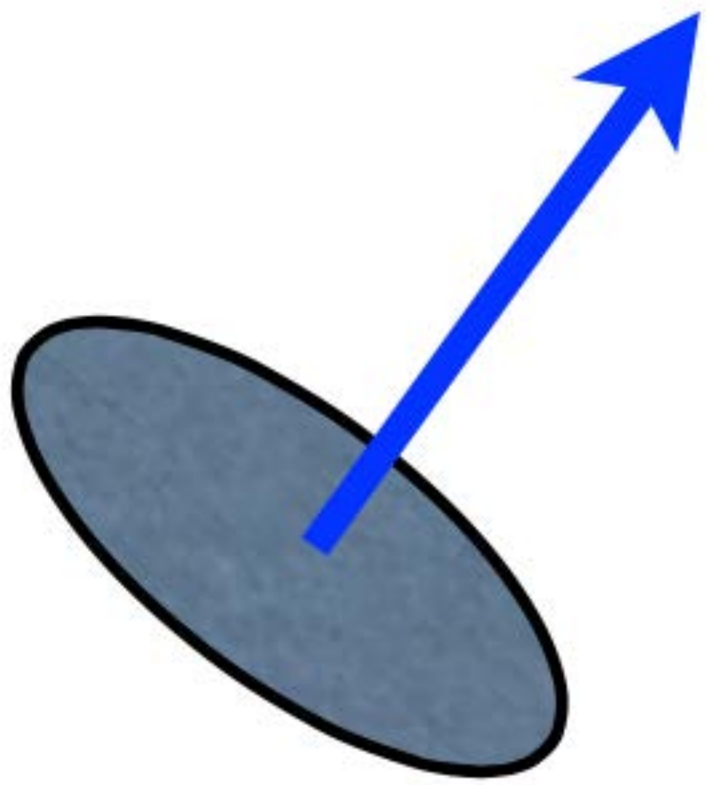
[redacted]



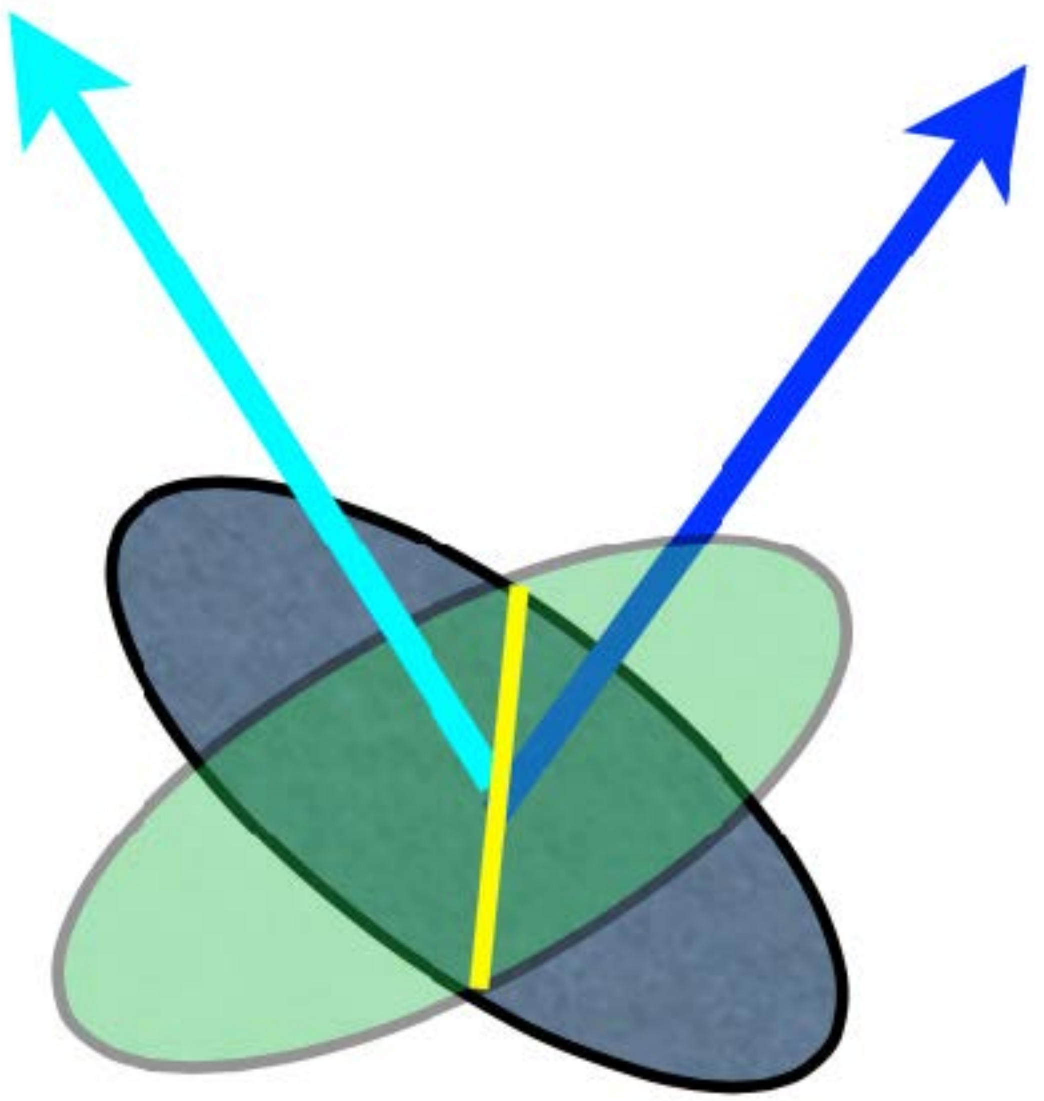
# Checkpoint




- How many unit vectors are perpendicular to a 2D vector?
    - 2 (positive and negative)
- How many unit vectors are perpendicular to a 3D vector?
    - Infinite and lie in plane



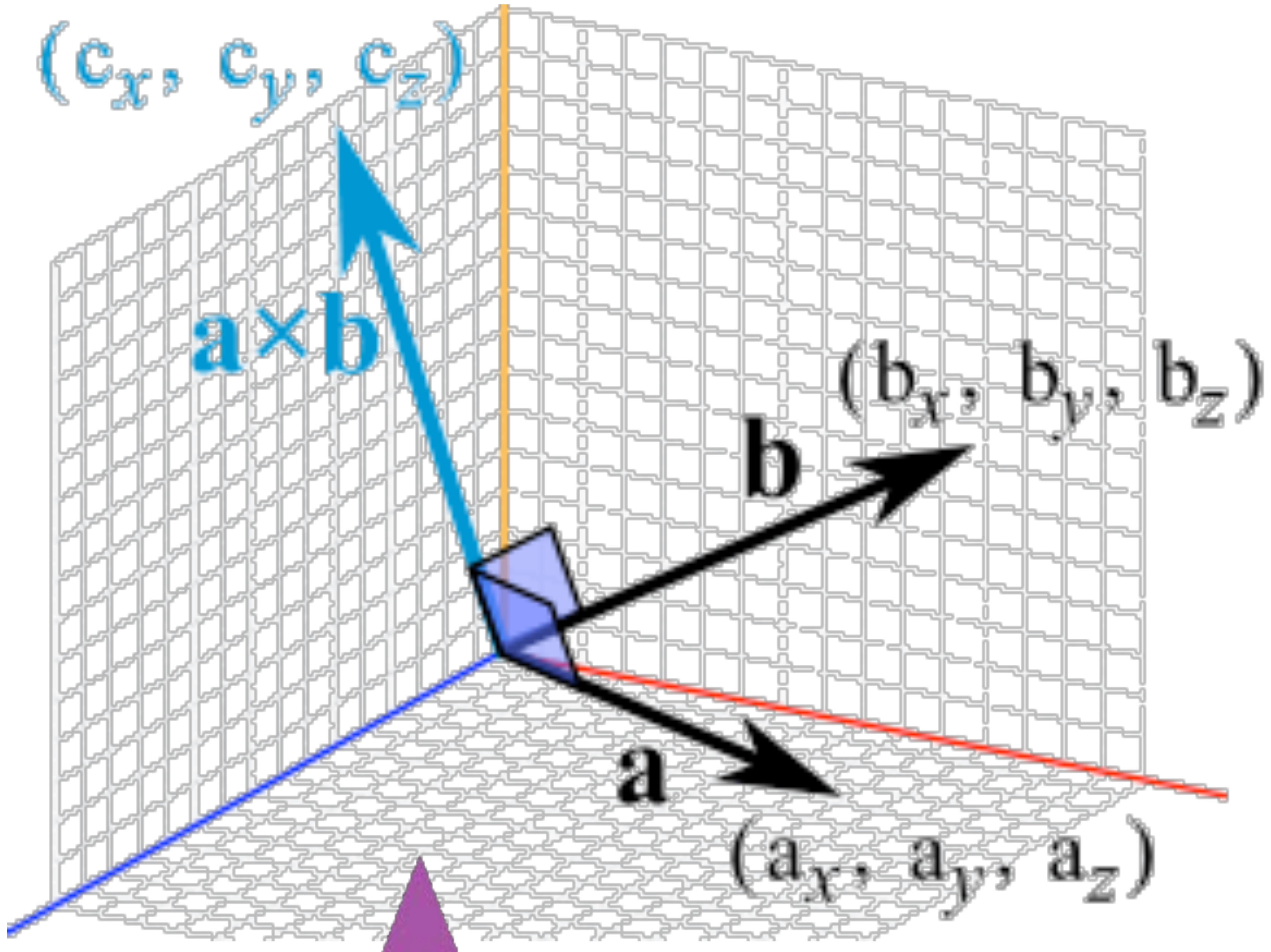
# Given two vectors, how to compute a vector orthogonal to both?



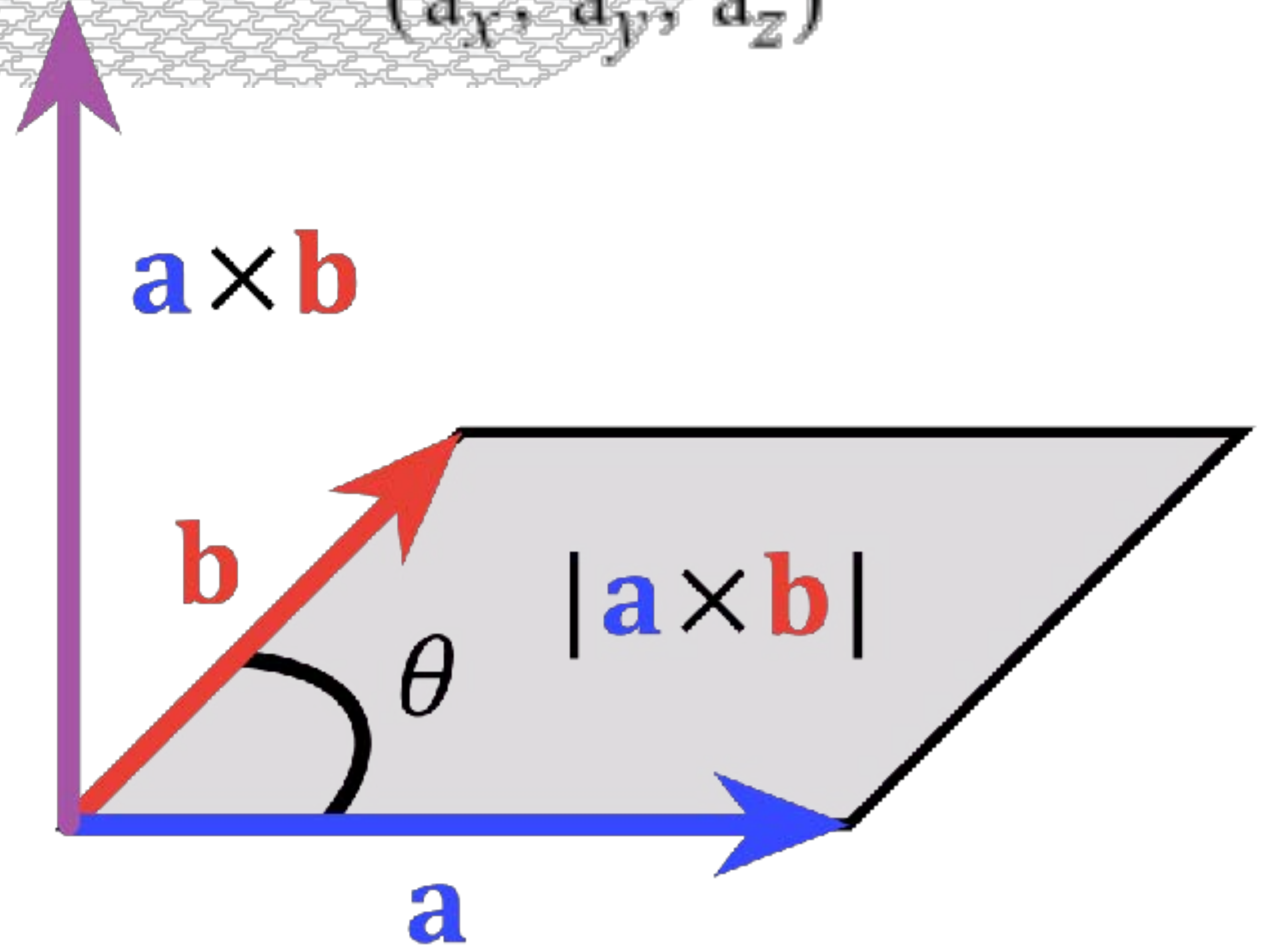
Assumes **a** and **b** are in same frame

# Cross Product

$$c_x = a_y b_z - a_z b_y$$
$$c_y = a_z b_x - a_x b_z$$
$$c_z = a_x b_y - a_y b_x$$




Results in new vector c orthogonal to both original vectors a and b




Length of vector c is equal to area of parallelogram formed by a and b

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \, \|\mathbf{b}\| \sin\theta$$

# Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [
    [1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0],
    [0, 0, 0, 1] ];
```

What is this matrix?

# Matrix-vector multiplication

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

# For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} =$$ [redacted]

# For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

# Matrix-vector multiplication

(two interpretations)

1) **Row story**: dot product of each matrix row

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix}$$

2) **Column story**: linear combination of matrix columns

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} j \\ k \\ l \end{bmatrix} = \begin{bmatrix} aj + bk + cl \\ dj + ek + fl \\ gj + hk + il \end{bmatrix} \qquad \begin{bmatrix} a \\ d \\ g \end{bmatrix} j + \begin{bmatrix} b \\ e \\ h \end{bmatrix} k + \begin{bmatrix} c \\ f \\ i \end{bmatrix} l$$

# Revisiting the cross product: Skew-symmetric matrices

A given 3D vector $\mathbf{a} = (a_1\ a_2\ a_3)^{\mathrm{T}}$

can be expressed as a skew-symmetric matrix

$$[\mathbf{a}]_\times = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}$$

such that the cross product with another vector is a matrix multiplication

$$\mathbf{a} \times \mathbf{b} = [\mathbf{a}]_\times \mathbf{b} .$$

# Linear Systems

We can use a variable instead of a vector, which gives us a linear system.

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Enabling the general form: $A\mathbf{x} = \mathbf{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\begin{aligned} a_{11}x_1 + \ & a_{12}x_2 + \cdots + \ a_{1n}x_n = b_1 \\ a_{21}x_1 + \ & a_{22}x_2 + \cdots + \ a_{2n}x_n = b_2 \\ \vdots \quad & \quad \vdots \qquad\qquad\quad \vdots \qquad \vdots \\ a_{m1}x_1 + \ & a_{m2}x_2 + \cdots + a_{mn}x_n = b_m. \end{aligned}$$

# Matrices

- A Matrix is a rectangular array of numbers

```
var mat = [
    [1, 0, 0, 0],
    [0, 1, 0, 0],
    [0, 0, 1, 0],
    [0, 0, 0, 1] ];
```

```
var mat = [
    [1, 0, 0, tx],
    [0, 1, 0, ty],
    [0, 0, 1, tz],
    [0, 0, 0, 1] ];
```

What is this matrix?

# Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} =$$ [redacted]

# Translation matrix example

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x + t_x \\ y + t_y \\ z + t_z \\ 1 \end{bmatrix}$$

# Matrix Multiplication

- Scalar Multiplication

$$\lambda\mathbf{A} = \lambda\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nm} \end{pmatrix} = \begin{pmatrix} \lambda A_{11} & \lambda A_{12} & \cdots & \lambda A_{1m} \\ \lambda A_{21} & \lambda A_{22} & \cdots & \lambda A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda A_{n1} & \lambda A_{n2} & \cdots & \lambda A_{nm} \end{pmatrix}.$$

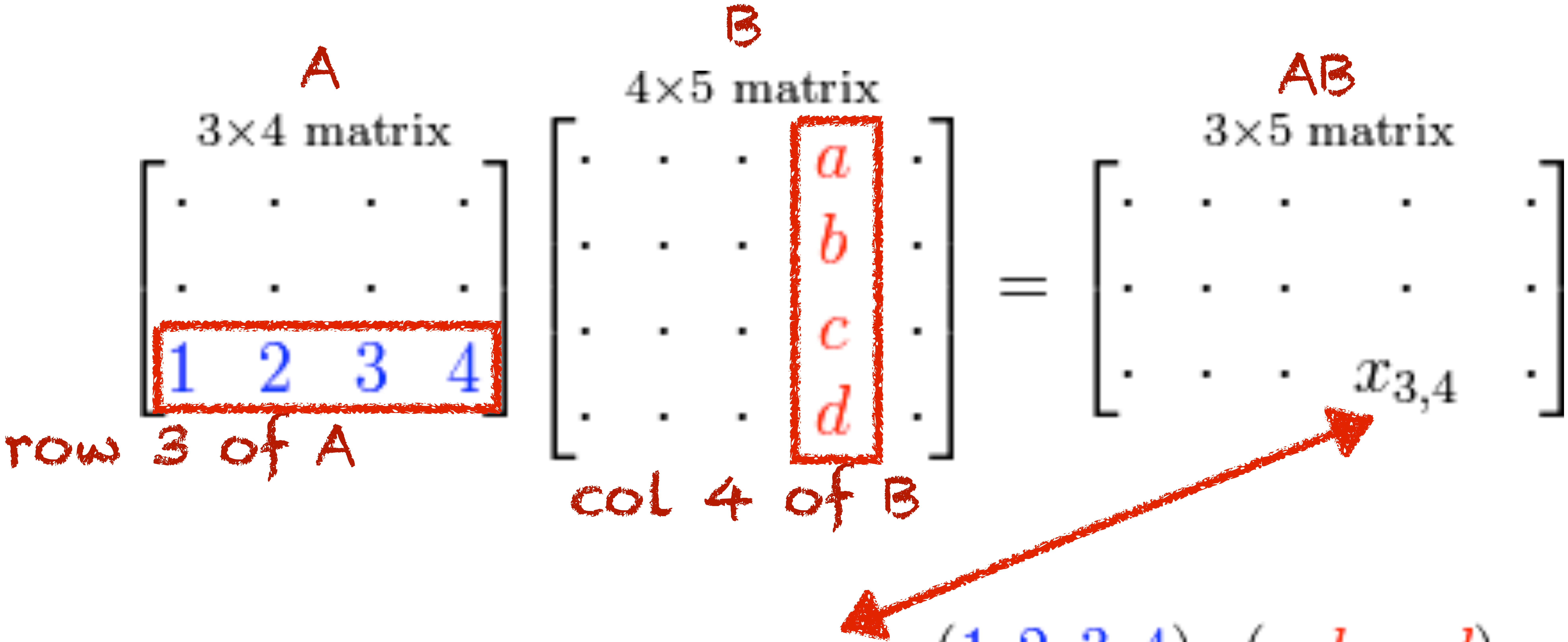
- Multiplication of two matrices

$$(\mathbf{AB})_{ij} = \sum_{k=1}^{m} A_{ik}B_{kj}\,.$$

Each entry of product matrix AB is a dot product of a row of A with a column of B

# Matrix multiplication

Finger sweeping rule should be second nature!
- Left finger sweeps left to right
- Right finger sweeps top to bottom



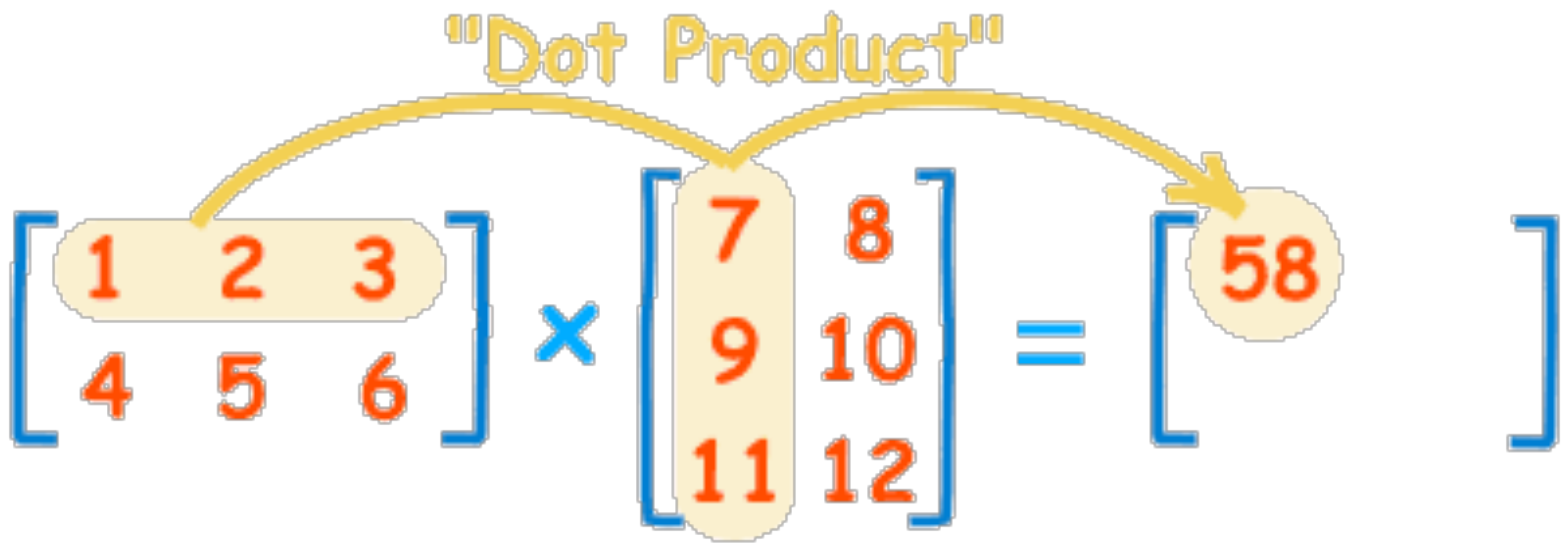

Do this dot product for each row/column combination

$$x_{3,4} = (1, 2, 3, 4) \cdot (a, b, c, d) = 1 \times a + 2 \times b + 3 \times c + 4 \times d$$

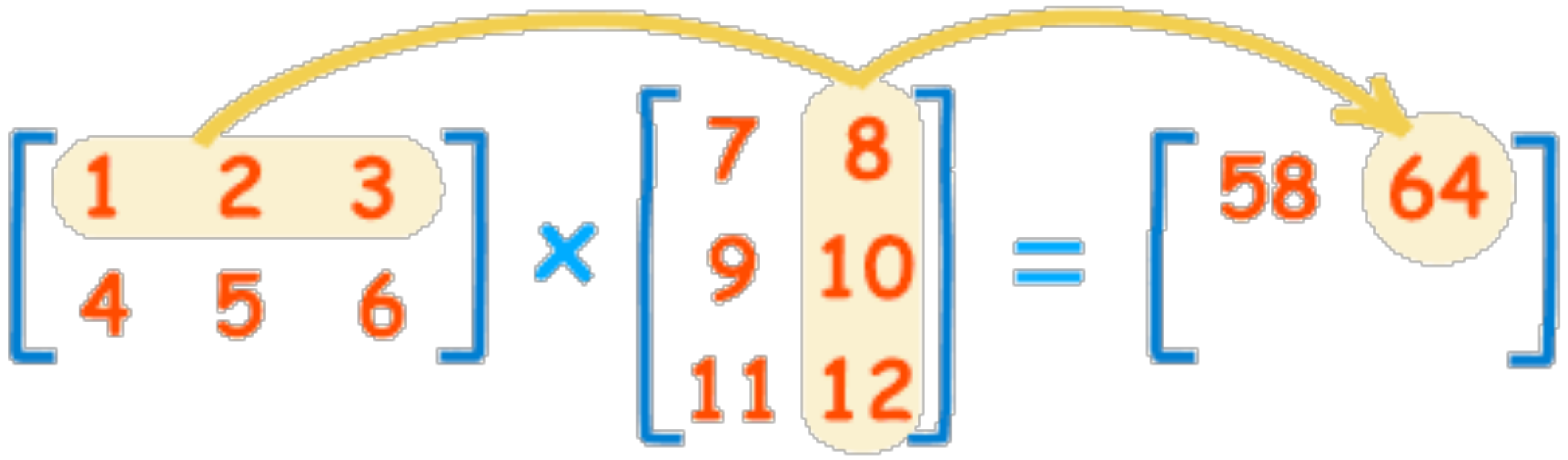
# Matrix Multiplication Reminders

- Number of columns of A must match number of rows of B
- Multiplying a (MxK) matrix with a (KxN) matrix will produce an (MxN) matrix
- Matrix multiplication is not commutative: AB != BA

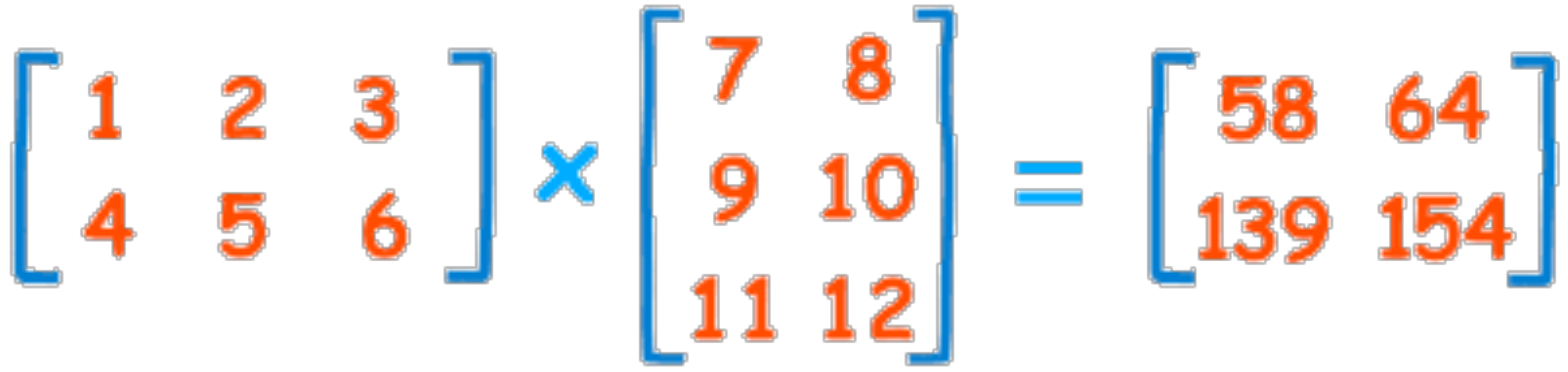
# Example




$(1, 2, 3) \bullet (7, 9, 11) = 1{\times}7 + 2{\times}9 + 3{\times}11 = 58$




$(1, 2, 3) \bullet (8, 10, 12) = 1{\times}8 + 2{\times}10 + 3{\times}12 = 64$




$(4, 5, 6) \bullet (7, 9, 11) = 4{\times}7 + 5{\times}9 + 6{\times}11 = 139$

$(4, 5, 6) \bullet (8, 10, 12) = 4{\times}8 + 5{\times}10 + 6{\times}12 = 154$

# For example

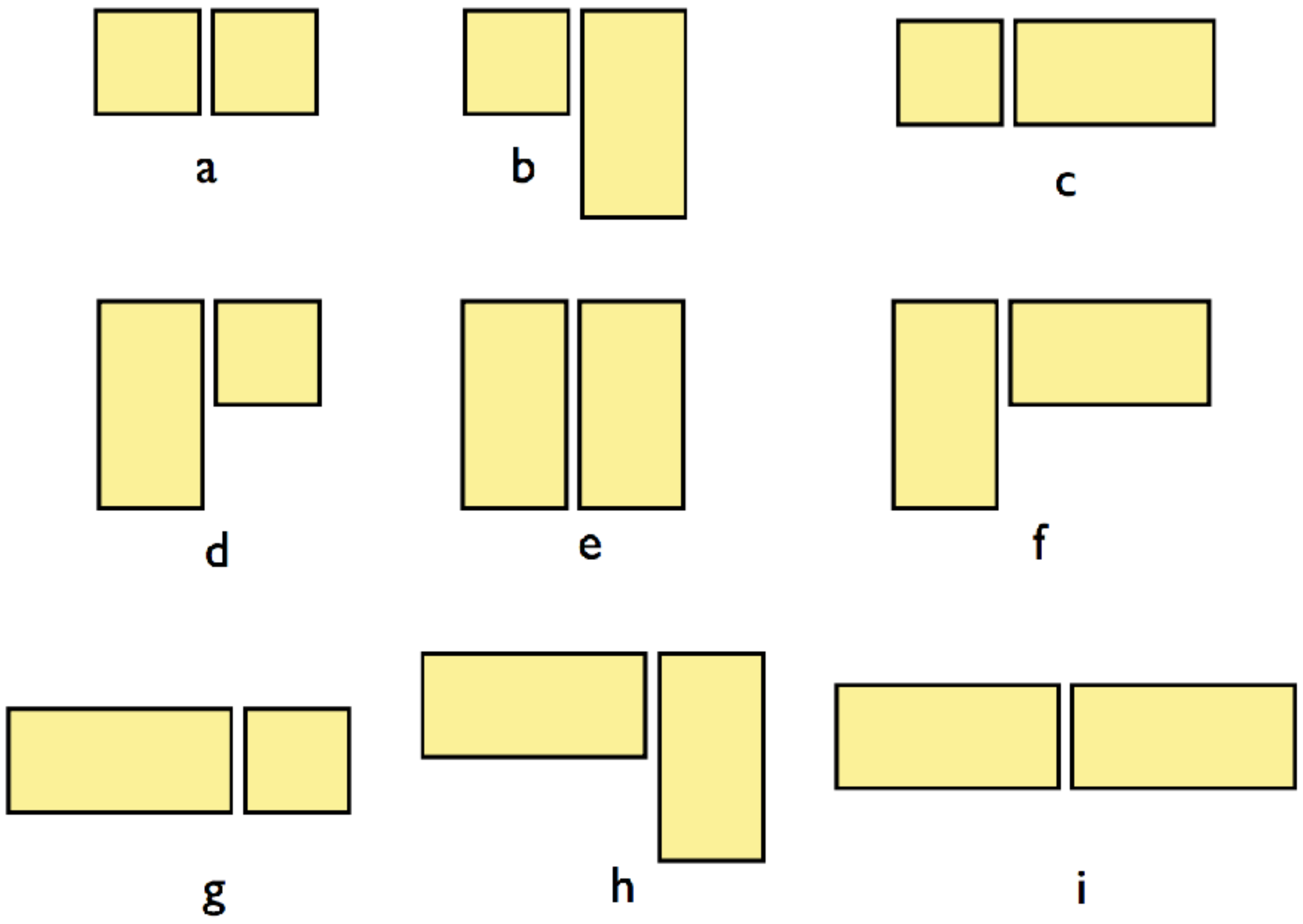
$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} =$$ [redacted]

# For example

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ 2 & 5 \end{bmatrix}$$

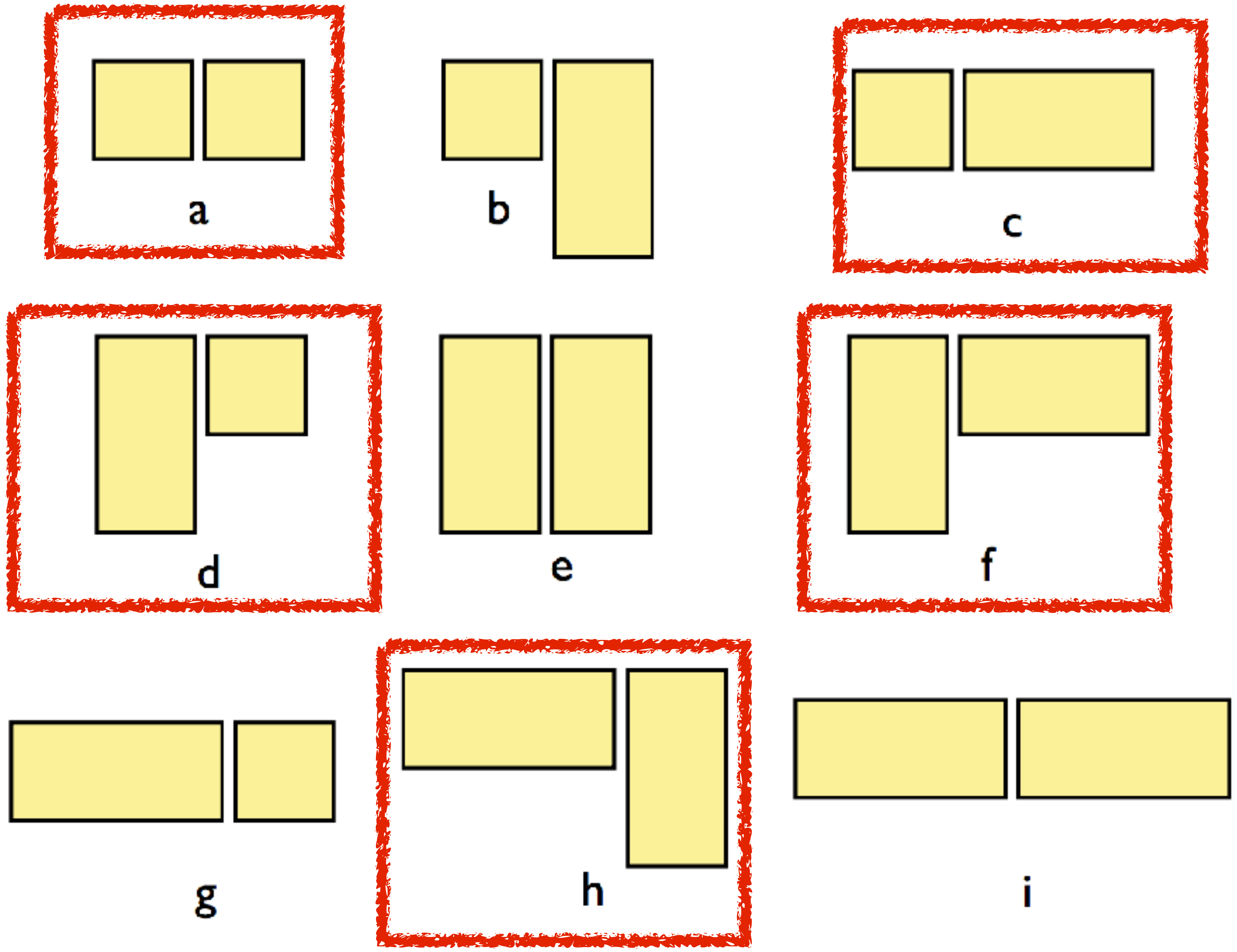
# Checkpoint

- Which of the following matrix multiplications are valid?



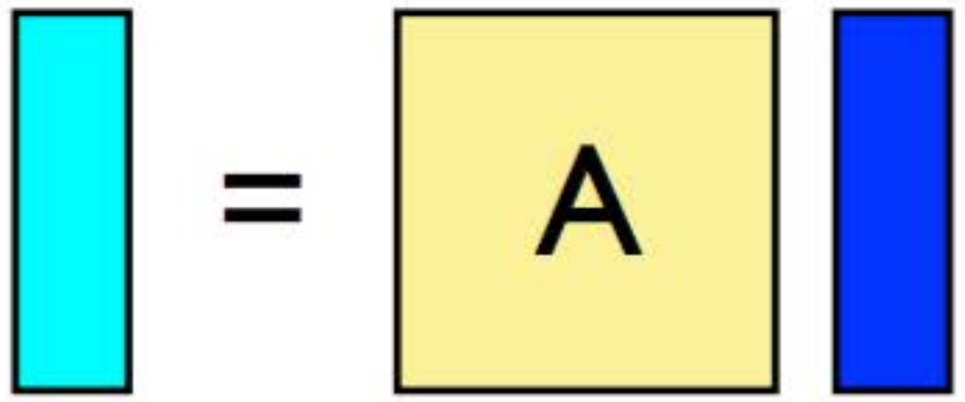
# Checkpoint

- Which of the following matrix multiplications are valid?



# Matrices as projections

- Matrix multiplication projects from one space to another.




Data projected into new coordinate system

Data in original coordinate system

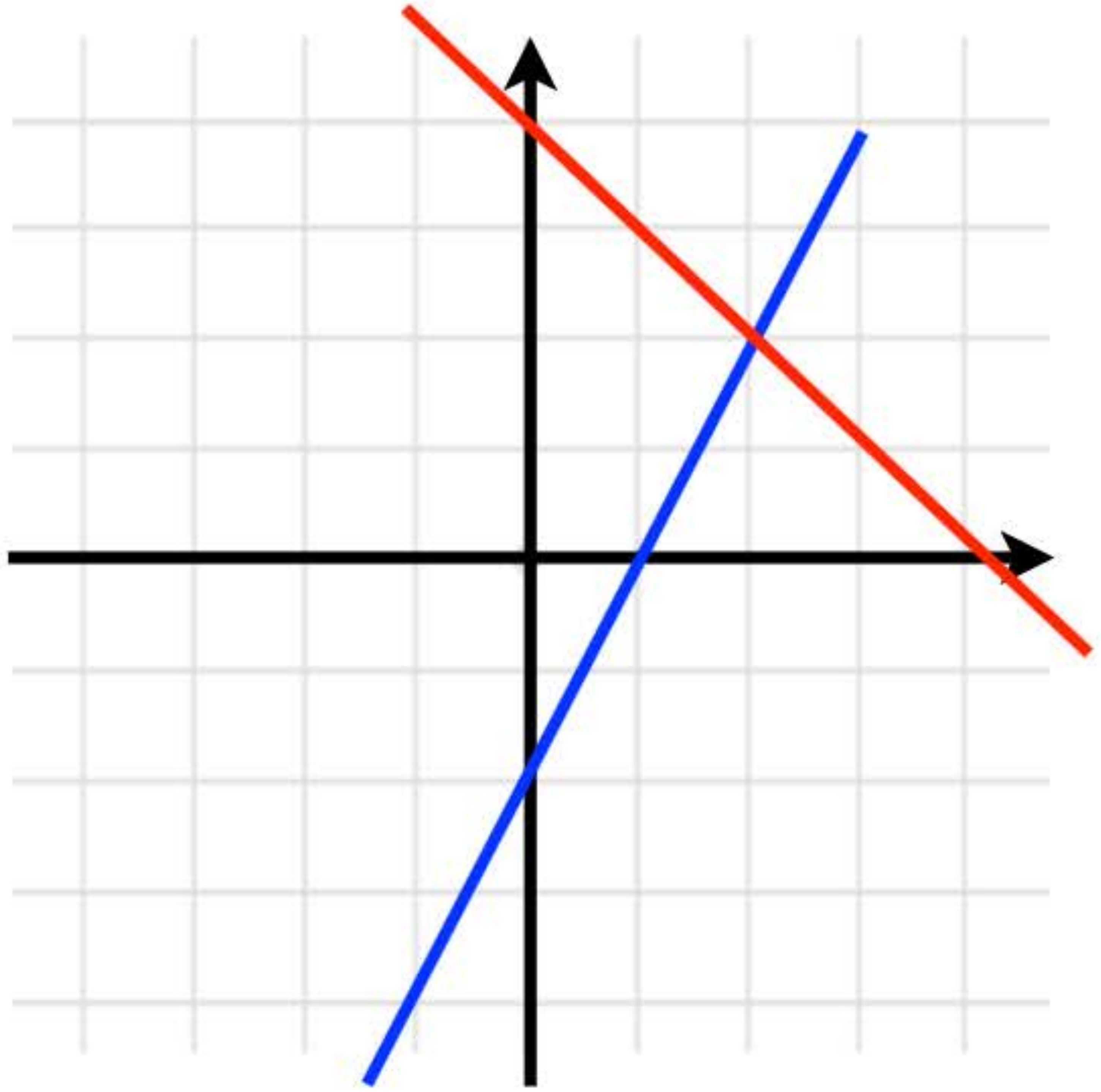
# Notable Matrices and Operations

- Matrix identity (I) causes no change: $A = I_m A = A I_n$
  - Diagonal elements $A_{ii} = 1$
  - Off-diagonal elements $A_{ij} = 0,\ i \neq j$
- Matrix inverse ($A^{-1}$): if $AA^{-1} = A^{-1}A = I$
- Distributing matrix inverse: $(AB)^{-1} = B^{-1}A^{-1}$
- Matrix transpose ($A^T$): a matrix's reflection about its diagonal
- Distributing matrix transpose: $(AB)^T = B^T A^T$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

# Matrix Geometry: Row Story

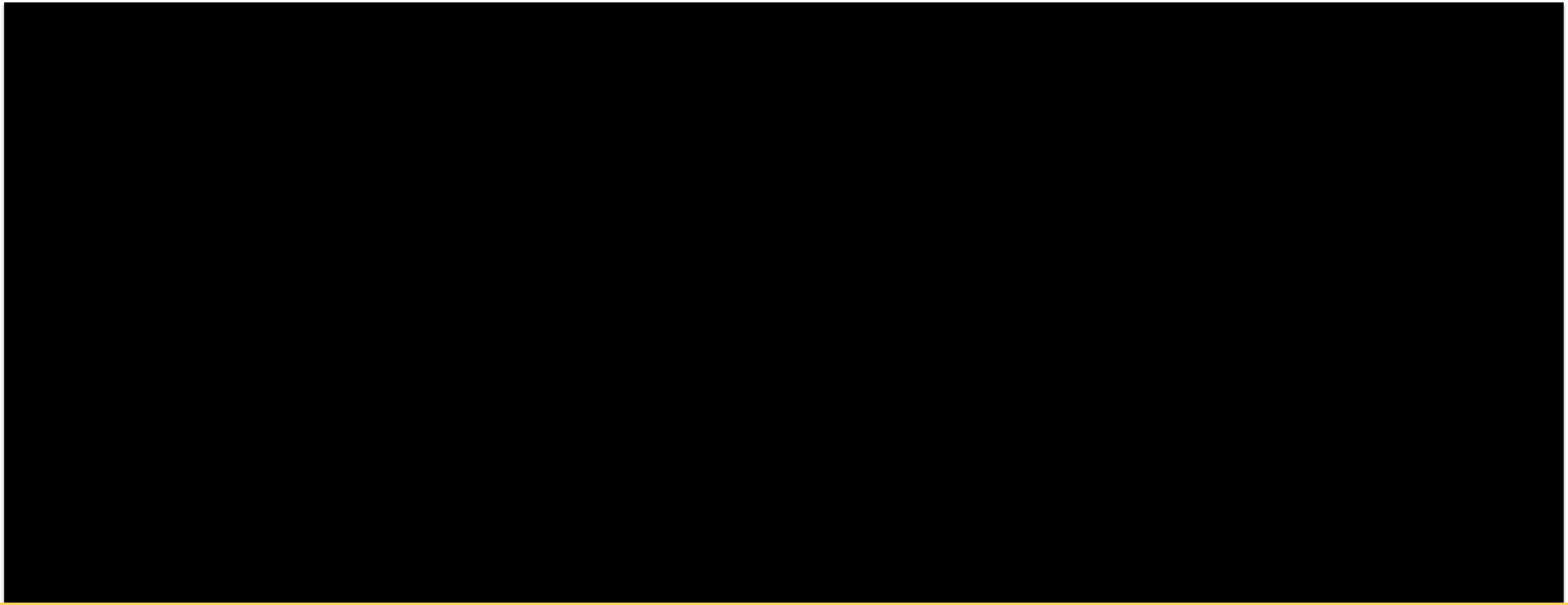


- Each row of a linear system represents a hyperplane. (In 2D, that's also a line!)
- The solution to the system is the intersection of those hyperplanes

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} x = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

# Solving linear systems

What would be the direct way to solve for ***x***? $$A\mathbf{x} = \mathbf{b}$$



# Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

# Matrix rank and inversion

- Let *A* be a square n by n matrix. *A* is invertible if full rank and a matrix *B* exists such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

- Rank of a matrix *A* is the size of the largest collection of linearly independent columns of *A*
- *A* is invertible (nonsingular) if it has full rank
- Gaussian elimination can find matrix inverse
- Singular matrix cannot be inverted this way

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$[I|B] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right]$$

# Solution by Decomposition

- In real applications, inverse not computed to solve linear systems
  - Efficiency, numerical precision, etc.
- Matrix decomposed into product of lower and upper triangular matrices
  - LU decomposition $A = LU$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

  - Cholesky decomposition $\mathbf{A} = \mathbf{L}\mathbf{L}^{\mathbf{T}}$
- Permits finding solution by forward substitution $\mathbf{L}\mathbf{y} = \mathbf{b}$ followed by backward substitution $\mathbf{L}^{\mathbf{T}}\mathbf{x} = \mathbf{y}$

# Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Can this always be done?

[redacted]

# Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Can this always be done?

No. But, we can approximate. How?

[redacted]

# Solving linear systems

What would be the direct way to solve for ***x***?

$$A\mathbf{x} = \mathbf{b}$$

Invert ***A*** and multiply by ***b***

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Can this always be done?

No. But, we can approximate. How?

Pseudoinverse least-squares approximation

$$\mathbf{x} = A^{+}_{\text{left}}\mathbf{b}$$

# Pseudoinverse

- For matrix A with dimensions N x M with full rank
- Find solution that minimizes squared error: $\|Ax - b\|_2$
- Left pseudoinverse, for when N > M, (i.e., "tall")

$$A_{\text{left}}^{-1} = \left(A^T A\right)^{-1} A^T \quad \text{s.t.} \quad A_{\text{left}}^{-1} A = I_n$$

- Right pseudoinverse, for when N < M, (i.e., "wide")

$$A_{\text{right}}^{-1} = A^T \left(AA^T\right)^{-1} \quad \text{s.t.} \quad AA_{\text{right}}^{-1} = I_m$$

# Next lecture:
# Representations I: Transformations